APM 236H1F term test 1

13 October, 2010

FAMILY NAME ____________________

GIVEN NAME(S) ____________________

STUDENT NUMBER ____________________

SIGNATURE ____________________

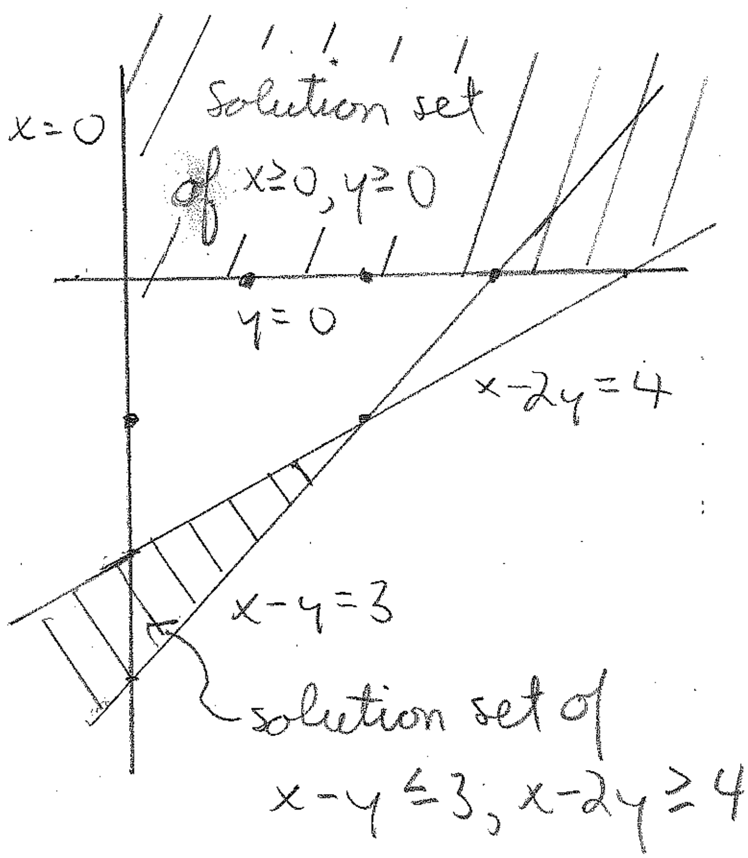
Instructions: No calculators or other aids allowed.

This test has 3 questions whose values are given immediately after the question numbers. Total marks = 40.

Write solutions in the spaces provided, using the backs of the pages if necessary. (Suggestion: If you have to continue a question, you may use the back of the **previous** page.) Aspects of any question which are indicated in **boldface** will be regarded as crucial during grading. **Show your work.**

The duration of this test is 50 minutes.

1. (13 marks) **Solve** the following problem **graphically**: Maximize $z = 5x + 6y$ subject to the constraints $\begin{array}{rcrcr} x & - & y & \leq & 3 \\ x & - & 2y & \geq & 4 \end{array}$, $x \geq 0, y \geq 0$.



The feasible region of the problem is the intersection of the two shaded regions. Since they do not intersect, the problem is <u>infeasible</u>.

2.(a) (4 marks) Which points in $\mathbb{R}^n$ belong to the **line segment** having endpoints $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^n$?

One correct answer is: points of the form $(1-\lambda)x_1 + \lambda x_2$ where $0 \leq \lambda \leq 1$

A second correct answer is: points of the form $c_1 x_1 + c_2 x_2$ where $c_1 + c_2 = 1$, $c_1 \geq 0$, $c_2 \geq 0$

2.(b) (5 marks) **Define** the term **convex set** (in $\mathbb{R}^n$).

One correct answer is: S is convex provided, for each x_1 and $x_2 \in S$, the line segment joining x_1 and x_2 lies in S.
A second correct answer is: S is convex provided, for each $x_1, x_2 \in S$ and $\lambda \in [0,1]$, $(1-\lambda)x_1 + \lambda x_2 \in S$.

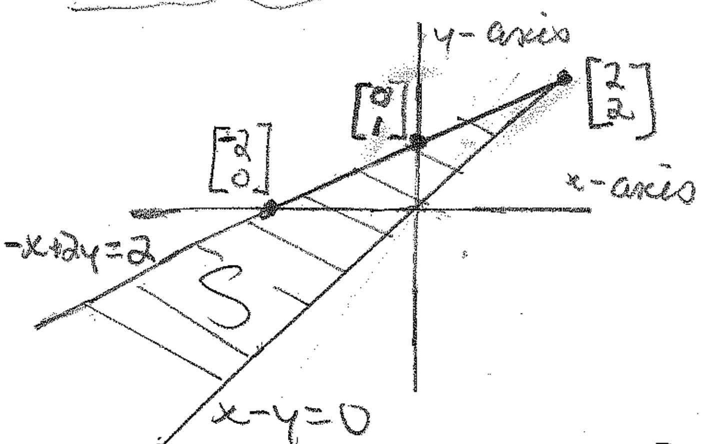
2.(c) (5 marks) Let $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \text{ s.t. } -x+2y \leq 2 \text{ and } x-y \leq 0 \right\}$. **Prove** that $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is **not an extreme point** of S.

$\begin{bmatrix} -2 \\ 0 \end{bmatrix} \in S$, since $-(-2)+2\cdot 0 \leq 2$ and $-2-0 \leq 0$

$\begin{bmatrix} 2 \\ 2 \end{bmatrix} \in S$, since $-2+2\cdot 2 \leq 2$ and $2-2 \leq 0$

Also, $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -2 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2 \\ 2 \end{bmatrix}$, yet $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} -2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

QED.



(This diagram is not part of the proof completed above. It was used to motivate the proof.)

3. (13 marks) Write **one** linear programming problem, $\mathcal{P}$, which satisfies **all** of the following:

(1) $\mathcal{P}$ has **2** decision variables, x and y.

(2) $\mathcal{P}$ is in **standard form.**

(3) The feasible region of $\mathcal{P}$ has $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as **extreme points.**

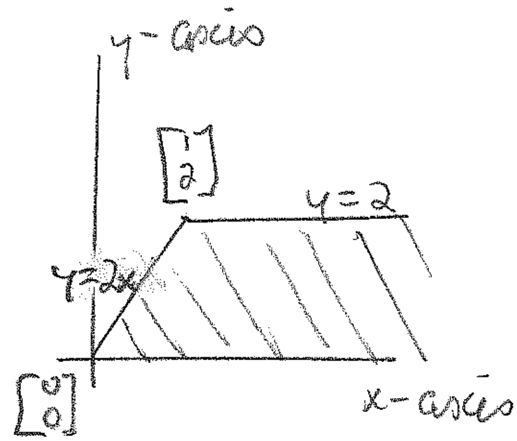
(4) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are the **only** extreme points of the feasible region of $\mathcal{P}$.

(5) $\mathcal{P}$ is **unbounded.**

One solution is:

Maximize $z = x$ s.t.

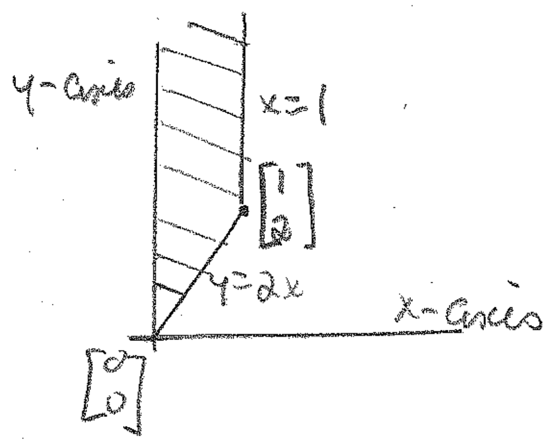
$$-2x + y \le 0$$
$$y \le 2$$
$$x \ge 0,\ y \ge 0$$



A second solution is:

Maximize $z = y$ s.t.

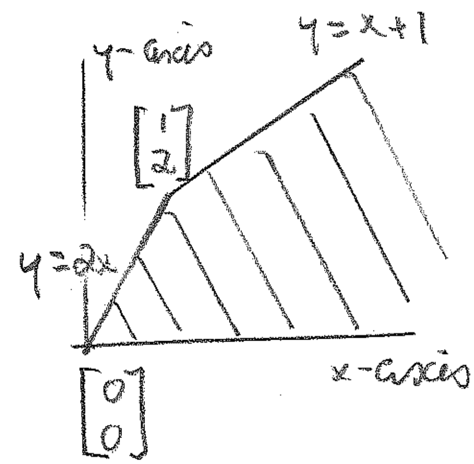
$$2x - y \le 0$$
$$x \le 1$$
$$x \ge 0,\ y \ge 0$$



A third solution is:

Maximize $z = x$ s.t.

$$-2x + y \le 0$$
$$-x + y \le 1$$
$$x \ge 0,\ y \ge 0$$



(Questions 2.(c) and 3 each have infinitely many solutions.)