MAT 137Y: Calculus!
Problem Set B

This problem set is intended to help you prepare for Test #2. It is not comprehensive: it only contains some problems that were not included in past problem sets or in past tutorials. You do not need to turn in any of these problems.

For further preparation review Problem Sets 3 and 4, Tutorials 5 through 8, and the practice problems from the textbook suggested on the course website (under "Videos and schedule").

1. Let f be a twice-differentiable function defined on an open interval I. Assume there are three points in the graph of f that are on the same line. Prove that there exists $c \in I$ such that $f''(c) = 0$.

   Hint: Draw a picture to understand what it means that 3 points in the graph of f are on the same line. Use MVT on two different intervals to conclude that there are two points where f' has the same value. Then use Rolle's Theorem.

2. Write a proof for the two theorems in Video 5.11 (at time 3:49).

3. The function $f(x) = 7 + 5x^3 + x^7$ is one-to-one. (You do not need to prove it). Compute $\left(f^{-1}\right)''(1)$.

4. For each of the following functions, find their local extrema, their global extrema, and the intervals where they are increasing or decresing:

   (a) $f(x) = x^{2/3}(2 - x)$ on the interval [-1,2].

   (b) $g(x) = |x^2 - 1|$ on the interval $[-2, 2]$.

5. Let $a \in \mathbb{R}$. Let f be a function which is differentiable at a. Assume that $f(a) > 0$.

   (a) Prove, using the definition of limit, that the function f must be positive near a. In other words, prove that there exists $\delta > 0$ such that $|x-a| < \delta \implies f(x) > 0$.

   (b) We define a new function g by $g(x) = \sqrt{f(x)}$. Prove that g is differentiable at a and that

   $$g'(a) = \frac{f'(a)}{2\sqrt{f(a)}}.$$

   Write a proof directly from the definition of derivative as a limit without using any of the other differentiation rules.

6. Prove that there is a constant C such that

$$\arcsin \frac{1-x}{1+x} + 2\arctan\sqrt{x} = C$$

for all x in a certain domain. What is the largest domain on which this identity is true? What is the value of the constant C?

7. Find all functions f that satisfy the following three properties:

   (a) The domain of f is $\mathbb{R}$.

   (b) f is differentiable everywhere.

   (c) f' is constant.

   The answer to this question should include two proofs. First, you will need to show that all the functions that you have found satisfy the three properties. Second, you will need to prove that *there are no other such functions*. This is the difficult part. To do this second proof, you need to assume that a function f satisfies the three properties, and prove that it must be one of the functions you listed. If your proof does not use Rolle theorem or the Mean Value Theorem (or a variant of them or a consequence of them or something equivalent) then it is probably wrong.