# MAT 137Y: Calculus!
## Problem Set A

This problem set is intended to help you prepare for Test #1. It is not comprehensive: it only contains some problems that were not included in past problem sets or in past tutorials. You do not need to turn in any of these problems.

For further preparation review Problem Sets 1 and 2, Tutorials 1 through 4, and the practice problems from the textbook suggested on the course website (under "Videos and schedule").

1. Let $f$ be some function for which you know only that

$$\text{if} \quad 0 < |x-3| < 1, \quad \text{then} \quad |f(x)-5| < 0.1.$$

Which of the following statements are necessarily true?

(a) If $|x-3| < 1$ and $x \neq 3$, then $|f(x)-5| < 0.1$.
(b) If $|x-3| < 1$, then $|f(x)-5| < 0.1$.
(c) If $|x-2.5| < 0.3$, then $|f(x)-5| < 0.1$
(d) $\lim_{x\to 3} f(x) = 5$.
(e) If $0 < |x-3| < 2$, then $|f(x)-5| < 0.1$.
(f) If $0 < |x-3| < 0.5$, then $|f(x)-5| < 0.1$.
(g) If $0 < |x-3| < \frac{1}{4}$, then $|f(x)-5| < \frac{1}{4}(0.1)$.
(h) If $0 < |x-3| < 1$, then $|f(x)-5| < 0.2$.
(i) If $0 < |x-3| < 2$, then $|f(x)-4.95| < 0.05$.
(j) If $\lim_{x\to 3} f(x) = L$, then $4.9 \leq L \leq 5.1$.

*Hint:* The answer is "true" for exactly five of the statements.

2. Given a real number $x$, we defined the *floor of* $x$, denoted by $\lfloor x \rfloor$, as the largest integer smaller than or equal to $x$. For example, $\lfloor \pi \rfloor = 3$, $\lfloor 7 \rfloor = 7$, and $\lfloor -0.5 \rfloor = -1$.

(a) At which points is the function $f(x) = \lfloor x \rfloor$ continuous?
(b) Consider the function $g(x) = \lfloor \sin x \rfloor$. Show that $g$ has exactly one removable and one non-removable discontinuity inside the interval $(0, 2\pi)$.

3. Write a formal proof for the "Continuity law for composition" (see video 2.16).

4. Write the formal definition of the following concepts:

(a) $\lim_{x\to a} f(x) = L$

(b) $\lim_{x\to a} f(x)$ exists

(c) $\lim_{x\to a} f(x) \neq L$

(d) $\lim_{x\to a} f(x)$ does not exist

(e) $\lim_{x\to a^+} f(x) = L$

(f) $\lim_{x\to a} f(x) = \infty$

(g) $\lim_{x\to a^-} f(x) = -\infty$

(h) $\lim_{x\to \infty} f(x) = L$

(i) $\lim_{x\to -\infty} f(x) = \infty$

*Note:* Between the videos, the textbook, and lectures you have learned some of them. You should be able to construct the rest (and many other variations) now by yourself. Note that the answers to Questions 4a and 4b are different, and the answers to Questions 4c and 4d are different as well.

5. Prove the following claims directly from the formal definitions.

(a) $\lim_{x\to 1} \frac{1}{x^2+1} = \frac{1}{2}$

(b) $\lim_{x\to 0} \frac{x}{|x|}$ does not exist

(c) $\lim_{x\to \infty} \frac{1}{x^2} = 0$

(d) $\lim_{x\to 1^+} \frac{1}{1-x} = -\infty$

6. Write a formal proof for the following theorem directly from the definition of limit (do not use any of the limit laws):

**Theorem** Let $f$ and $g$ be functions with domain $\mathbb{R}$. Let $L, M \in \mathbb{R}$.

- IF $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to\infty} g(x) = M$
- THEN $\lim_{x\to\infty} [f(x) - g(x)] = L - M$.