MAT137

- Problem set 3 due next Wednesday
- Test 2 is at 2:00pm on June 20
- Today's Topic: Differentiation
- **Watch 3.13-3.20, 4.1, 4.2 before Friday**

True or False

Let $a \in \mathbb{R}$.
Let f be a function with domain $\mathbb{R}$.
Assume f is differentiable everywhere.
What can we conclude?

1. $f(a)$ is defined.
2. $\lim_{x \to a} f(x)$ exists.
3. f is continuous at a.
4. $f'(a)$ exists.
5. $\lim_{x \to a} f'(x)$ exists.
6. f' is continuous at a.

A continuity lemma

Write a formal proof for:

Lemma

Let $a \in R$.
Let g be a function continuous at a.
Assume that $g(a) \neq 0$. Then $g(x) \neq 0$ for x close to a.

Note: First, figure out what "$g(x) \neq 0$ for x close to a" means formally.

Write a formal proof for the quotient rule for derivatives

Theorem

- Let $a \in \mathbb{R}$.
- Let f and g be functions defined at and near a. Assume $g(a) \neq 0$.
- We define the function h by $h(x) = \dfrac{f(x)}{g(x)}$.

IF f and g are differentiable at a,

THEN h is differentiable at a, and

$$h'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

Write a proof directly from the definition of derivative.
Hint: Imitate the proof of the product rule in Video 3.6.

Check your proof

- Are there words or only equations?
- Does every step follow logically?
- Did you only assume things you could assume?

- At some point in your proof you must have used, for example, that g was continuous. (Otherwise your proof is most likely wrong.)
  Did you notice you were using this? Did you justify it?

Critique this proof – #2

$$h'(x) = \lim_{y \to x} \frac{h(y) - h(x)}{y - x} = \lim_{t \to x} \frac{\frac{f(y)}{g(y)} - \frac{f(x)}{g(x)}}{y - x}$$

$$= \ldots \quad \text{(assume the rest of the algebra is here)}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

When $x = a$: $\quad h'(a) = \dfrac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$

Critique this proof – #1

Note: We did not do this question in class

$$h'(a) = \lim_{x\to a} \frac{h(x)-h(a)}{x-a} = \lim_{x\to a} \frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x-a}$$

$$= \lim_{x\to a} \frac{f(x)g(a) - f(a)g(x)}{g(x)g(a)\,(x-a)}$$

$$= \lim_{x\to a} \frac{f(x)g(a) - f(a)g(a) + f(a)g(a) - f(a)g(x)}{g(x)g(a)\,(x-a)}$$

$$= \lim_{x\to a} \left\{ \left[\frac{f(x)-f(a)}{x-a} g(a) - f(a) \frac{g(x)-g(a)}{x-a} \right] \frac{1}{g(x)g(a)} \right\}$$

$$= [f'(a)g(a) - f(a)g'(a)] \frac{1}{g(a)g(a)}$$

A pesky function

Let $h(x) = x^2 \sin \frac{1}{x}$, where we define $h(0) = 0$.

1. Calculate $h'(x)$ for any $x \neq 0$.
2. Using the definition of derivative, calculate $h'(0)$.
3. Is h continuous at 0?
4. Is h differentiable at 0?
5. Is h' continuous at 0?

Hint: The last two questions have different answers.

Computations!

Compute the derivatives of the following functions:

- $f(x) = \tan(3x^2 + 1)$
- $f(x) = (\cos x)(\sin 2x)(\tan 3x)$
- $f(x) = \cos(\sin(\tan x))$
- $f(x) = \cos\left(3x + \sqrt{1 + \sin^2 x^2}\right)$

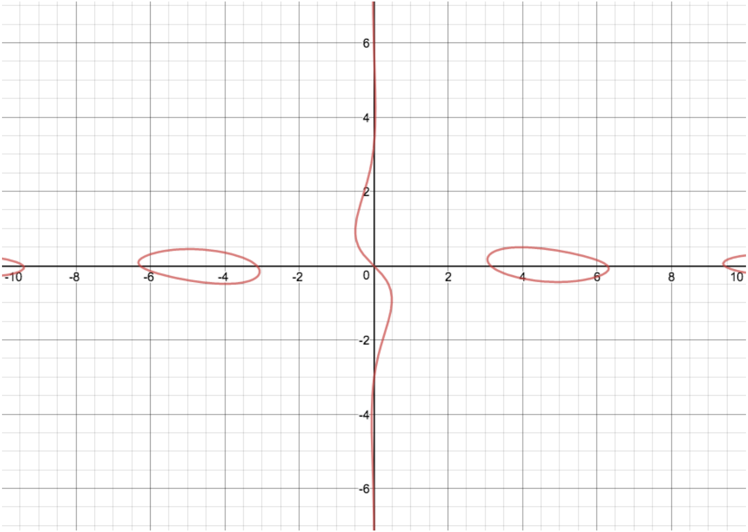
The equation

$$\sin(x+y) + xy^2 = 0$$

defines a function $y = h(x)$ near $(0,0)$. ▸ graph

Using implicit differentiation, compute

1. $h(0)$
2. $h'(0)$
3. $h''(0)$
4. $h'''(0)$


6
4
2
0
-2
-4
-6
-10
-8
-6
-4
-2
2
4
6
8
10

Derivatives of $(f \circ g)$

Note: We did not do this question in class, but it is a good exercise

Assume f and g are functions that have all their derivatives.

Find formulas for

1. $(f \circ g)'(x)$
2. $(f \circ g)''(x)$
3. $(f \circ g)'''(x)$

in terms of the values of f, g and their derivatives.

Hint: The first one is simply the chain rule.

Computations

Note: We did not do this question in class, but it is a good exercise

Compute the derivative of

1. $f(x) = \sqrt{2x^2 + x + 1}$
2. $f(x) = \sqrt{x + \sqrt{x + \sqrt{x + 1}}}$

From the derivative to the function

1. Sketch the graph of a continuous function whose derivative has the graph below
2. Sketch the graph of a non-continuous function whose derivative has the graph below