MAT137

- Term Test 1 is on Monday at 6-8pm
  - More information is posted on Quercus

- Extra office hours: 11-1pm on Friday
- TA office hours: 4-6pm on Friday

- Today's Topic: Continuity and limits

- **Watch 2.21, 2.22, 3.1-3.5 before Friday**

Critique this "proof" – #1

- $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies |f(x)g(x)| < \varepsilon$.
- $\forall \varepsilon_1 > 0, \exists \delta_1 > 0$ s.t. $0 < |x - a| < \delta_1 \implies |f(x)| < \varepsilon_1$
- $\exists M > 0$ s.t. $\forall x \neq 0, |g(x)| \leq M$
- $|f(x)g(x)| = |f(x)||g(x)| < \varepsilon_1 M$
- $\varepsilon = \varepsilon_1 M \implies \varepsilon_1 = \dfrac{\varepsilon}{M}$
- $\delta = \delta_1$

Critique this "proof" – #2

- WTS: $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies |f(x)g(x)| < \varepsilon$.
- Let $\varepsilon > 0$.
- We know $\lim_{x \to a} f(x) = 0$. In this definition, let $\varepsilon_1 = \dfrac{\varepsilon}{M}$.
- We know $\exists \delta_1 \in \mathbb{R}$ s.t. $0 < |x - a| < \delta_1 \implies |f(x)| < \varepsilon_1 = \dfrac{\varepsilon}{M}$.
- Assume $0 < |x - a| < \delta$
- Since $\exists M > 0$ s.t. $\forall x \neq 0, |g(x)| \leq M$
  $|f(x)g(x)| \leq \dfrac{\varepsilon}{M} \cdot M = \varepsilon$.

Critique this "proof" – #3

- Since g is bounded, $\exists M > 0$ s.t.$\forall x \neq 0, |g(x)| \leq M$
- Since $\lim_{x \to a} f(x) = 0$, there exists $\delta_1 > 0$ s.t. if $0 < |x - a| < \delta_1$, then $|f(x) - 0| = |f(x)| < \varepsilon_1 = \dfrac{\varepsilon}{M}$.

$$|f(x)g(x)| = |f(x)| \cdot |g(x)| \leq |f(x)| \cdot M < \varepsilon_1 \cdot M = \frac{\varepsilon}{M} \cdot M = \varepsilon$$

- In summary, by setting $\delta = \min\{\delta_1\}$, we find that if $0 < |x - a| < \delta$, then $|f(x) \cdot g(x)| < \varepsilon$.

Undefined function

Let $a \in \mathbb{R}$ and let f be a function. Assume $f(a)$ is undefined.

What can we conclude?

1. $\lim_{x \to a} f(x)$ exist
2. $\lim_{x \to a} f(x)$ doesn't exist.
3. No conclusion. $\lim_{x \to a} f(x)$ may or may not exist.

What else can we conclude?

4. f is continuous at a.
5. f is not continuous at a.
6. No conclusion. f may or may not be continuous at a.

True or False?

(Assuming these limits exist)

$$\lim_{x \to a} g(f(x)) = g\left(\lim_{x \to a} f(x)\right)$$

Fix

What extra condition do we need to add for this to be true?

$$\lim_{x \to a} g(f(x)) = g\left(\lim_{x \to a} f(x)\right)$$

A difficult example

Note: We did not do this question in class

Is it possible to construct functions such that....?

1. $\lim_{x \to 1} f(x) = 2$
2. $\lim_{u \to 2} g(u) = 3$
3. $\lim_{x \to 1} g(f(x)) = 42$

New continuous functions

We want to prove the following theorem

Theorem

IF f and g are continuous functions
THEN $h(x) = \max\{f(x), g(x)\}$ is also a continuous function.

You are allowed to use all results that we already know. What is the fastest way to prove this?

Hint: What is the number $\frac{a+b+|a-b|}{2}$?
There is a way to prove this quickly without writing any epsilons.

A question from last year's test

The only thing we know about the function g is that

$$\lim_{x \to 0} \frac{g(x)}{x^2} = 2.$$

Use it to compute the following limits (or explain that they do not exist):

1. $\lim_{x \to 0} \frac{g(x)}{x}$
2. $\lim_{x \to 0} \frac{g(x)}{x^4}$
3. $\lim_{x \to 0} \frac{g(3x)}{x^2}$

Which solution is right?

Compute $L = \lim_{x \to -\infty} \left[x - \sqrt{x^2 + x}\right]$.

- **Solution 1**

$$L = \lim_{x \to -\infty} \frac{\left[x - \sqrt{x^2 + x}\right] \left[x + \sqrt{x^2 + x}\right]}{\left[x + \sqrt{x^2 + x}\right]} = \lim_{x \to -\infty} \frac{x^2 - (x^2 + x)}{\left[x + \sqrt{x^2 + x}\right]}$$

$$= \lim_{x \to -\infty} \frac{-x}{x\left[1 + \sqrt{1 + \frac{1}{x}}\right]} = \lim_{x \to -\infty} \frac{-1}{\left[1 + \sqrt{1 + \frac{1}{x}}\right]} = \frac{-1}{2}$$

- **Solution 2**

$$L = \lim_{x \to -\infty} \left[x - \sqrt{x^2 + x}\right] = (-\infty) - \infty = -\infty$$

Rational limits

Note: We did not do this question in class

Compute:

1. $\lim\limits_{x \to -3^+} \dfrac{x^2 - 9}{3 - 2x - x^2}$
2. $\lim\limits_{x \to 1^+} \dfrac{x^2 - 9}{3 - 2x - x^2}$

Computations!

Using that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, compute the following limits:

1. $\lim_{x\to 2} \frac{\sin x}{x}$
2. $\lim_{x\to 0} \frac{\sin(5x)}{x}$
3. $\lim_{x\to 0} \frac{\tan^2(2x^2)}{x^4}$
4. $\lim_{x\to 0} \frac{\sin e^x}{e^x}$
5. $\lim_{x\to 0} \frac{1-\cos x}{x}$
6. $\lim_{x\to 0} \frac{\tan^{10}(2x^{20})}{\sin^{200}(3x)}$
7. $\lim_{x\to 0} \left[(\sin x)\,(\cos(2x))\,(\tan(3x))\,(\sec(4x))\,(\csc(5x))\,(\cot(6x))\right]$

Key takeaway: How do you know when to use $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$?

Limits at infinity

Note: We did not do this question in class - but it will be on test 1

Compute:

1. $\lim_{x\to\infty} \left(x^7 - 2x^5 + 11\right)$
2. $\lim_{x\to\infty} \left(x^2 - \sqrt{x^5+1}\right)$
3. $\lim_{x\to\infty} \frac{x^2+11}{x+1}$
4. $\lim_{x\to\infty} \frac{x^2+2x+3}{3x^2+4x+5}$
5. $\lim_{x\to\infty} \frac{x^3+\sqrt{2x^6+1}}{2x^3+\sqrt{x^5+1}}$
6. $\lim_{x\to\infty} \arctan(x)$
7. $\lim_{x\to\infty} \frac{\sin(x)}{x}$