MAT137

- Problem Set 2 is due **Tuesday** May 21
  - This is so we have time to mark the problem sets before the term test

- Monday/Tuesday tutorials are cancelled due to Victoria Day

- Today's Topic: Limits and continuity

- **Watch 2.14 - 2.20 before Wednesday**
  Watch 2.21, 2.22, 3.1-3.5 before next Friday

Preparation: choosing deltas

1. Find a value of $\delta > 0$ such that

$$|x - 3| < \delta \implies |5x - 15| < 1.$$

2. Find *all* value of $\delta > 0$ such that

$$|x - 3| < \delta \implies |5x - 15| < 1.$$

3. Find a value of $\delta > 0$ such that

$$|x - 3| < \delta \implies |5x - 15| < 0.1.$$

4. Let us fix $\varepsilon > 0$. Find a value of $\delta > 0$ such that

$$|x - 3| < \delta \implies |5x - 15| < \varepsilon.$$

Your first $\varepsilon - \delta$ proof

Goal

We want to prove that

$$\lim_{x \to 3} (5x + 1) = 16 \tag{1}$$

directly from the definition.

1. Write down the formal definition of the statement (1).
2. Write down what the structure of the formal proof should be, without filling the details.
3. Write down a complete formal proof.

Your second $\varepsilon - \delta$ proof

Goal

We want to prove that

$$\lim_{x \to 2} \left(x^2 - 4x + 5\right) = 1 \tag{2}$$

directly from the definition.

1. Write down the formal definition of the statement (2).
2. Write down what the structure of the formal proof should be, without filling the details.
3. Rough work: What is δ?
4. Write down a complete formal proof.

A harder proof

Goal

We want to prove that

$$\lim_{x \to 0} \left(x^3 + x^2\right) = 0 \tag{3}$$

directly from the definition.

1. Write down the formal definition of the statement (3).
2. Write down what the structure of the formal proof should be, without filling the details.
3. Rough work: What is δ?
4. Write down a complete formal proof.

Is this proof correct?

Claim:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \quad 0 < |x| < \delta \implies |x^3 + x^2| < \varepsilon.$$

Proof.

- Let $\varepsilon > 0$.
- Take $\delta = \sqrt{\frac{\varepsilon}{|x+1|}}$.
- Let $x \in \mathbb{R}$. Assume $0 < |x| < \delta$. Then

$$|x^3 + x^2| = x^2|x+1| < \delta^2|x+1| = \frac{\varepsilon}{|x+1|}|x+1| = \varepsilon.$$

- I have proven that $|x^3 + x^2| < \varepsilon$.

□

Indeterminate form

Let $a \in \mathbb{R}$.
Let f and g be functions defined near a.
Assume $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$.

What can we conclude about $\lim_{x \to a} \frac{f(x)}{g(x)}$?

1. The limit is 1.
2. The limit is 0.
3. The limit is ∞.
4. The limit does not exist.
5. We do not have enough information to decide.

True or False?

Is this theorem true?

Claim

Let $a \in \mathbb{R}$.

Let f and g be functions defined near a.

- IF $\lim_{x \to a} f(x) = 0$,
- THEN $\lim_{x \to a} [f(x)g(x)] = 0$.

True or False?

Is this statement true or false?

Claim

Let f be a function with domain $\mathbb{R}$.

If $\lim_{x \to a} |f(x)| = 0$, then $\lim_{x \to a} f(x) = 0$.

Limit Computations

Calculate the following limits or explain why they don't exist.

1. $\lim_{x \to \frac{\pi}{2}} \frac{\cos(x)}{x}$
2. $\lim_{x \to \infty} \frac{\sin(x)}{x}$
3. $\lim_{x \to -\infty} \frac{7x^3 + \sin(x)}{14 - x^3}$
4. $\lim_{x \to \infty} \frac{\sin(5x^2) + \cos(2x+3)\sin(8x-4)}{x^2 - 3}$
5. $\lim_{x \to 0} x^3 e^{\cos \frac{1}{x}}$

A new theorem about products

Theorem

Let $a \in \mathbb{R}$. Let f and g be functions with domain $\mathbb{R}$, except possibly a. Assume

- $\lim_{x \to a} f(x) = 0$, and
- g is bounded. This means that

$$\exists M > 0 \text{ s.t. } \forall x \neq a, |g(x)| \leq M.$$

THEN $\lim_{x \to a} [f(x)g(x)] = 0$

1. Write down the formal definition of what you want to prove.
2. Write down what the structure of the formal proof.
3. Rough work.
4. Write down a complete formal proof.

Critique this "proof" – #1

- $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x-a| < \delta \implies |f(x)g(x)| < \varepsilon$.
- $\forall \varepsilon_1 > 0, \exists \delta_1 > 0$ s.t. $0 < |x-a| < \delta_1 \implies |f(x)| < \varepsilon_1$
- $\exists M > 0$ s.t. $\forall x \neq 0, |g(x)| \leq M$
- $|f(x)g(x)| = |f(x)||g(x)| < \varepsilon_1 M$
- $\varepsilon = \varepsilon_1 M \implies \varepsilon_1 = \dfrac{\varepsilon}{M}$
- $\delta = \delta_1$

Critique this "proof" – #2

Note: We did not do this question in class

- WTS: $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies |f(x)g(x)| < \varepsilon$.
- Let $\varepsilon > 0$.
- We know $\lim_{x \to a} f(x) = 0$. In this definition, let $\varepsilon_1 = \dfrac{\varepsilon}{M}$.
- We know $\exists \delta_1 \in \mathbb{R}$ s.t. $0 < |x - a| < \delta_1 \implies |f(x)| < \varepsilon_1 = \dfrac{\varepsilon}{M}$.
- Assume $0 < |x - a| < \delta$
- Since $\exists M > 0$ s.t. $\forall x \neq 0, |g(x)| \leq M$
  $|f(x)g(x)| \leq \dfrac{\varepsilon}{M} \cdot M = \varepsilon$.

Critique this "proof" – #3

Note: We did not do this question in class

- Since g is bounded, $\exists M > 0$ s.t.$\forall x \neq 0, |g(x)| \leq M$
- Since $\lim_{x \to a} f(x) = 0$, there exists $\delta_1 > 0$ s.t. if $0 < |x - a| < \delta_1$, then $|f(x) - 0| = |f(x)| < \varepsilon_1 = \dfrac{\varepsilon}{M}$.

$$|f(x)g(x)| = |f(x)| \cdot |g(x)| \leq |f(x)| \cdot M < \varepsilon_1 \cdot M = \frac{\varepsilon}{M} \cdot M = \varepsilon$$

- In summary, by setting $\delta = \min\{\delta_1\}$, we find that if $0 < |x - a| < \delta$, then $|f(x) \cdot g(x)| < \varepsilon$.