MAT137

- Problem set 4 due next Wednesday

- **If you have a conflict with Test 2, you have to email me by June 13**

- Today's Topic: Rolle's theorem and Mean Value Theorem

- **Watch 6.1-6.10 before next Wednesday**
  Watch 6.11-6.16 before next Friday

Zeroes of the derivative

Construct a function f that is differentiable on $\mathbb{R}$ and such that

1. f has exactly 2 zeroes and f' has exactly 1 zero.
2. f has exactly 2 zeroes and f' has exactly 2 zeroes.
3. f has exactly 3 zeroes and f' has exactly 1 zero.
4. f has exactly 1 zero and f' has infinitely many zeroes.

A new theorem

We want to prove this theorem:

Theorem 1

Let f be a differentiable function defined on an interval I.
IF $\forall x \in I, f'(x) \neq 0$
THEN f is one-to-one on I.

1. Transform $[P \implies Q]$ into $[(\text{not } Q) \implies (\text{not } P)]$.
   You get an equivalent Theorem (call it "Theorem 2").
   We are going to prove Theorem 2 instead.
2. Write the definition of "f is not one-to-one on I". You will need it.
3. Recall the statement of Rolle's Theorem. You will need it.
4. Do some rough work if needed.
5. Write a complete proof for Theorem 2.

A variant

What should go in the blanks to make this theorem true?

Theorem 3

Let $a < b$. Let f be a function defined on $[a, b]$.
IF

- f is continuous on ______
- f is differentiable on ______
- f is not one-to-one on $[a, b]$

THEN $\exists c \in (a, b)$ such that $f'(c) = 0$.

Why the three hypotheses are necessary

Theorem 3

Let $a < b$. Let f be a function defined on $[a, b]$.
IF

1. f is continuous on $[a, b]$
2. f is differentiable on (a, b)
3. f is not one-to-one on $[a, b]$

THEN $\exists c \in (a, b)$ such that $f'(c) = 0$.

Give three examples to justify that each of the three hypotheses are necessary for the theorem to be true. (Graphs of the examples are enough).

Complete the statement for this theorem and prove it.

Rolle's Theorem 2

Let $a < b$. Let f be a function defined on $[a.b]$.
IF

- (Some conditions on continuity and derivatives)
- $f(a) = f'(a) = 0$
- $f(b) = 0$

THEN $\exists c \in (a, b)$ such that $f''(c) = 0$.

Hint: Apply the 1st Rolle's Theorem to f, then do something else.

The N-th Theorem of Rolle

Note: We did not do this question in class but it is a good exercise

Complete the statement for this theorem and prove it.

Rolle's Theorem N

Let N be a positive integer.
Let $a < b$. Let f be a function defined on $[a.b]$.
IF

- (Some conditions on continuity and derivatives)
- (Some conditions at a)
- $f(b) = 0$

THEN $\exists c \in (a, b)$ such that $f^{(N)}(c) = 0$.

Positive derivative implies increasing

Use the MVT to prove

Theorem

Let $a < b$. Let f be a differentiable function on (a, b).

- IF $\forall x \in (a, b), f'(x) > 0$,
- THEN f is increasing on (a, b).

1. Recall the definition of what you are trying to prove.
2. From that definition, figure out the structure of the proof.
3. If you have used a theorem, did you verify the hypotheses?
4. Are there words in your proof, or just equations?

What is wrong with this proof?

Theorem

Let $a < b$. Let f be a differentiable function on (a, b).

- IF $\forall x \in (a, b), f'(x) > 0$,
- THEN f is increasing on (a, b).

Proof.

- From the MVT, $f'(c) = \dfrac{f(b) - f(a)}{b - a}$
- We know $b - a > 0$ and $f'(c) > 0$
- Therefore $f(b) - f(a) > 0$, so $\quad f(b) > f(a)$
- f is increasing.

□

Cauchy's MVT - Part 1

Here is a new theorem:

We want to prove this Theorem

Let $a < b$. Let f and g be functions defined on $[a, b]$.
IF (some conditions)
THEN $\exists c \in (a, b)$ such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$

What is wrong with this "proof"?

- By MVT, $\exists c \in (a, b)$ s.t. $f'(c) = \dfrac{f(b) - f(a)}{b - a}$
- By MVT, $\exists c \in (a, b)$ s.t. $g'(c) = \dfrac{g(b) - g(a)}{b - a}$
- Divide the two equations and we get what we wanted.

$\square$

Cauchy's MVT - Part 2

We want to prove this theorem

Let $a < b$. Let f and g be functions defined on $[a, b]$.
IF (some conditions)
THEN $\exists c \in (a, b)$ such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$

1. There is one number $M \in \mathbb{R}$ so that you will be able to apply Rolle's Theorem to the new function $H(x) = f(x) - Mg(x)$ on the interval $[a, b]$. What is M?
2. Apply Rolle's Theorem to H. What do you conclude?
3. Fill in the missing hypotheses in the theorem above.
4. Prove it.

Counterexample from last class

Consider the functions:

$$f_1(x) = x^2 \qquad f_2(x) = x^2$$
$$g_1(x) = \sqrt{x} \qquad g_2(x) = e^x$$
$$f_1(g_1(x)) = (\sqrt{x})^2 \qquad f_2(g_2(x)) = (e^x)^2$$

Which of these functions are injective?

Question: Why are the compositions injective even though f function is not injective?
(*Hint*: are you restricting the domain of f?)

Proving difficult identities

Prove that, for every $x \geq 0$,

$$\arcsin \frac{1-x}{1+x} + 2 \arctan \sqrt{x} = \frac{\pi}{2}$$

Hint: Take derivatives.

Intervals of monotonicity

Let $g(x) = x^3(x^2 - 4)^{1/3}$.

Find out on which intervals this function is increasing or decreasing.
Using that information, sketch its graph.

To save time, here is the first derivative:

$$g'(x) = \frac{x^2(11x^2 - 36)}{3(x^2 - 4)^{2/3}}$$

Note: My graph in class was wrong, you should double check your answer using by graphing the function with a graphing software

Your first integration

Note: We did not do this question in class but it is a good exercise

Find all functions f such that, for all $x \in \mathbb{R}$:
$f''(x) = x + \sin x$.

Inequalities

Note: We did not do this question in class but it is a good exercise

Prove that, for every $x \in \mathbb{R}$

$$e^x \geq 1 + x$$

Hint: When is the function $f(x) = e^x - 1 - x$ increasing or decreasing?