SYMPLECTIC CAPACITY AND CONVEXITY

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1. Symplectic Capacities

Gromov's nonsqueezing theorem shows that the radii of balls and cylinders are stored somehow as a symplectic invariant. In particular, a ball and cylinder of the same radius are in some way the same by virtue of one just barely being able to embed in the other. It would be nice to extend this invariant to larger classes of sets. Unfortunately, *a priori* we can reflect this invariant in two apparently distinct ways.

From the first point of view, the ball "recognizes" the size of the cylinder by being able or unable, depending on its radius, to symplectically embed within it. Therefore the *cylindrical capacity* of a set is defined in the following way:

Let $Z^{2n}(r)$ be the standard symplectic cylinder of radius r. Then for any non-empty subset U of a $2n$-dimensional symplectic manifold,

$$c^Z(U) := \inf\{\pi r^2 \mid \exists \psi \ s.t.\ \psi(U) \subset Z^{2n}\}$$

where ψ is a symplectomorphism.

From the second point of view, the cylinder "recognizes" the size of the ball by, conversely, being able or unable to accomodate it, depending on its radius. The *symplectic radius* of a set is defined as follows:

$B^{2n}(r)$ is the standard symplectic ball of radius r. With U as before,

$$c_B(U) := \sup\{\pi r^2 \mid \exists \psi \ s.t.\ \psi(B^{2n}(r)) \subset U\}$$

where, again, we are in the symplectic category.

These two functions are obviously symplectic invariants. However, as feared, for all but a small class of sets, they will not be equal. In fact, there is a large class of functions which equally well reflect the original invariant quantity we hoped to capture.

Definition 1.1. A *symplectic capacity* c is a map from all symplectic manifolds (M, ω) to the non-negative reals such that

(1) Monotonicity — If (M, ω) embeds symplectically in (N, τ) then $c(M, \omega) \leq c(N, \tau)$.
(2) Conformality — If α is a non-zero real, $c(M, \alpha\omega) = |\alpha| c(M, \omega)$.
(3) Normalization — $c(B^{2n}(r)) = c(Z^{2n}(r)) = \pi r^2$.

There are many different symplectic capacities, but it follows from the motonicity and normalization properties, as well as the nonsqueezing theorem, that c_B and c^Z are the minimal and maximal capacities respectively. Because of bounds like this, in spite of the non-uniqueness of symplectic capacities, they share enough in common, and provide enough information, that one can use them and their existence generically to get some strong results about, for example, the topology of the space of symplectomorphisms. In particular, they provide a cleaner proof of Gromov and Eliashberg's theorem about the closure of the symplectmorphisms in the space of diffeomorphisms. (Is Siddarth talking about this?)

2. Volume and Convexity

If symplectic capacities could be related in some way to, for example, Riemannian invariants such as volume, then they could be much better understood. However, as I understand it, symplectic capacity is fundamentally a 2-dimensional quantity, and not just in that it scales like r^2. Generic objects of fixed capacity ought to be understood as having some flexibilty in all but two of their dimensions, though in precisely what sense is not clear to me. Therefore certainly capacity cannot be a function of volume.

However, we can ask such questions as: for which objects is the ratio between capacity and volume maximal; for which is it minimal? These questions were studied by Claude Viterbo. Furthermore, if these questions could be answered in general, then they could serve as an intermediate step to give constraints on the relationship between *different* capacities. So while there is no hope of getting a unique capacity, there is a notion of equivalence of capacities, which holds for c_1 and c_2 if there exists a constant κ such that

$$\kappa^{-1}c_1 \leq c_2 \leq \kappa c_1,$$

where equivalence of c_B and c^Z implies equivalence of all others.

Of course, the cylinder is an example of a set for which the capcity-volume ratio is minimal—it has finite capacity but infinite volume. However, in the maximal case, it turns out that there are sets of arbitrarily small volume for any given capacity (and fixed dimension). Therefore a restricted class of sets is considered, in which case a reasonable conjecture is possible.

We will state Viterbo's conjecture, but for now leave unspecified which class of sets it ought to apply to:

Conjecture 2.1 (Viterbo)**.** *For any symplectic capacity, among all sets of class X in the standard symplectic space* $\mathbb{R}^{2n}$ *with a given volume, the ball has maximum capacity.*

In other words, for any set U in class X,

$$\frac{c(U)}{c(B^{2n})} \leq \left(\frac{Vol(K)}{Vol(B^{2n})}\right)^{1/n}$$

First, we should note that even though the conjecture is saying something about the symplectic capacity of the ball, it is actually the cylinder we would work with in any attempt at a proof: since we *know* the capacity of the ball, the conjecture really bounds the capacity of the set U from above, and the only way of doing that is by embedding it in a cylinder. Therefore a proof seeks to embed sets of given volume into cylinders as small as possible.

A simple class of sets for which the conjecture is known to hold is the L^p-balls. A simple example of sets for which the conjecture fails in a very strong sense are constructed out of star-shaped cross-sections. (See diagram?) There is a symplectic capacity known as *displacement energy*, which, for a subset of a symplectic space, measures how much of an energy difference a Hamiltonian function must have so that its Hamiltonian flow displaces the set from itself entirely in a unit amount of time. If one has satisfied oneself that this is indeed a capacity, then one recognizes that a star shape exploits it: a star can be narrowed to have arbitrarily small volume but still have a minimum necessary displacement to avoid overlap.

The appropriate class X appears to be the convex sets. Why would one think this? Perhaps by looking at the counterexample, or perhaps one could be inspired by the discussion of ellipsoids in the next section. However, convexity is not a symplectic invariant, whereas the inequality certainly is. So of course we are talking about sets which are symplectomorphic to convex sets in $\mathbb{R}^{2n}$. It is unclear if there is a class more natural to the symplectic category for which conjectures of this sort ought to hold, but most of the research so far has focused on proving the convex case.

3. Progress Toward the Theorem

The full conjecture has not yet been proven. What Viterbo proved originally was as follows: that there exist γ_n depending on n such that, for any convex set K,

$$\frac{c(U)}{c(B^{2n})} \leq \gamma_n \left(\frac{Vol(K)}{Vol(B^{2n})} \right)^{1/n},$$

and in particular $\gamma_n \leq 32n$ or possibly less in the case of sufficiently symmetric bodies. This follows a common pattern in mathematics in which the full conjecture may be reinterpreted as the statement that $\gamma_n = 1$ for all n.

Since then, progress has been made on the constants γ. An earlier paper by Artstein-Avidan and Ostrover bounds γ_n by $A \log^2 n$ for a universal constant A. The most recent result constrains γ_n to be a constant independent of n but still possibly greater than 1. We present some of the ideas in the argument:

Among the most well-behaved sets from the point of view of symplectic capacity are ellipsoids. As the diagram illustrates, ellipsoids for which the two shortest axes are equal, i.e. for which the smallest bisecting cross-section is a circle, have a unique symplectic capacity by virtue of the fact that they may be sandwiched between a ball and cylinder of equal radius. Therefore it is not unreasonable to suppose that ellipses could be used as building blocks in a proof of Viterbo's conjecture.

There is a sense in which ellipses are archetypes of convex sets. However, that sense is only an approximate one: for any convex set of any dimension, there is an ellipse which is not too much larger and not too much smaller. One precise statement (among many) that can be made is the following:

Theorem 3.1 (Milman)**.** *There exists a universal constant C such that for any convex set K of any dimension there exists an ellipsoid $\mathcal{E}_K$ such that*

(1) $Vol(\mathcal{E}_K) = Vol(K)$
(2) $Vol(K + \mathcal{E}_K)^{1/n} \leq C\, Vol(K)^{1/n}$
(3) $Vol(K \cap \mathcal{E}_K)^{1/n} \geq C^{-1}\, Vol(K)^{1/n}$

where $K + \mathcal{E}_K$ is the Minkowski sum.

Such an ellipsoid is called an M-ellipsoid, after Vitali Milman. These sorts of results fall in what is called *asymptotic geometric analysis.* (Shall I draw a picture?)

The value of C can be strengthened (i.e. brought closer to 1) by imposing different amounts of symmetry on the convex set K, however we should not expect C to be 1. In particular, taking $K + \mathcal{E}_K$ should certainly enlarge the set, but the point is that it can be done in a controllable way, independent of the dimension or any other factor.

Milman goes on to characterize in detail these M-ellipsoids. Since they are not unique, and are less so if the constant C is relaxed, there is some freedom in choosing them, and this freedom will clearly be needed if they need to be compatible with the symplectic structure.

Artstein-Avidan, Milman and Ostrover use these M-ellipsoids to prove their result:

Theorem 3.2. *There exists a universal constant C such that*

$$\frac{c(U)}{c(B^{2n})} \leq C \left(\frac{Vol(K)}{Vol(B^{2n})} \right)^{1/n}$$

However, their argument does not find capacities directly from the ellipsoid associated to each convex set, but instead achieves the result indirectly. Milman's theorem is employed as an intermediate step to give a further approximate relation, not just between a convex set and its Minkowski sum with an M-ellipsoid, but also between the volume of two convex sets taken in Minkowski sum, and the sum of their volumes taken indpendently. As a technical detail, before the approximation holds, these sets might first need be subject to a suitable linear transformation, roughly speaking to put their shapes in line (precisely speaking, to put their M-ellipsoids in line). (Draw a picture??)

An argument follows which takes any convex set and symmetrizes it with respect to the complex structure on $\mathbb{R}^{2n}$, using the approximation to control the growth of the volume. Then the complex structure is used to pick out an appropriate complex plane. The orthogonal projection of the set onto this plane is shown to have a constrained radius, and therefore a suitably small enveloping cylinder can be extruded from it.

4. The Nonlinear Category? and Fancy

Artstein-Avidan, Milman and Ostrover express some surprise, at least from the symplectic-geometric point of view, that a result apparently so close to the full conjecture can be achieved using only linear embeddings, commenting that in symplectic geometry one typically needs highly nonlinear objects to get strong results. The particular approximations they use make it seem unlikely that anything too similar to their approach will get C down to 1.

However, if the linear symplectic category has gotten within a constant of the full conjecture, then one can speculate that it could get one all the way. If this is not the case, it's not clear how the leap to the nonlinear category will be made. A (presumably highly uneducated) picture of how it might work has struck me. I hope to spend a little bit of time in the near future investigating it:

A surface or volume integral over the convex body defines a function on $\mathbb{R}^{2n}$ which is used to select a parametrized family of constant-surfaces. This function would be designed so that, first of all, for generic convex bodies of a suitable degree of asymmetry, each such surface has the topology of $S^1 \times \mathbb{R}^{2n-1}$, i.e. the boundary of a cylinder, and, second of all, that this topological cylinder will be symplectomorphic to the standard cylinder.

Perhaps he first property might be achieved by integrating a kernel which decays rapidly in directions transverse to the surface of the body, so that the elongated axis of the body is represented at large in $\mathbb{R}^{2n}$ as a long spike in the function carrying out to infinity.

Perhaps the second property might be explicitly achieved by constructing a Hamiltonian function through a second integral over the body. Nonetheless, the surfaces must still be chosen in the first place so that they really do bound symplectic cylinders!

The hope is that as one takes cylinders at progressively higher values of the defining function, they become optimally tight about the body, and one can use some calculus to recognize this. Of course I am allowing for the likelihood that the cylinder which actually achieves contact with the body will be degenerate in some way.

As it stands this is pure fancy, and would require expertise either to dismiss or to make work.

References

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