Introductory Topics in Symplectic Geometry

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If you find mistake / typos / suspicious points, please let me know at
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Preface

These notes accompany my course "Introduction to Symplectic Geometry" at the University of Toronto in Winter Term 2018–19. The course runs over 12 weeks, with three 50-minute lectures per week.

Here is the formal course description:

> This is an introductory course in symplectic geometry and topology. We will discuss a variety of concepts, examples, and theorems, which may include, but are not restricted to, these topics: Moser's method and Darboux's theorem; Hamiltonian group actions and momentum maps; almost complex structures and holomorphic curves; Gromov's nonsqueezing theorem.
> Prerequisites: Manifolds and differential forms; homology.

An informal goal is to use the course topics as an excuse to reinforce the prerequisites.

Chapters roughly correspond to weeks, and sections roughly correspond to lectures. This correspondence is only approximate; I will often arrange, or rearrange, these notes differently from the lectures. But—at least for the duration of this course—I will try to be consistent with the numbering of the exercises: exercises are numbered consecutively within each chapter, and the numbered exercises in each chapter are the ones that are assigned on the corresponding week and are generally due on the following week.

Certainly I will not be able to write up these notes at the pace that the term progresses. The best possible scenario would be that I produce reasonable titles with occasional partial content. Without commitment, I will try to at least include those exercises that I would like students who are taking the course for credit to submit in writing. These will be the exercises in the text that are numbered.

Jesse Frohlich has kindly agreed to post his lecture notes on the course website. Hopefully he will be more consistent than me.

Please let me know of any mistakes / typos / suspicious points that you find in these notes.

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CHAPTER 0

Review

Here are quick reviews of manifolds and differential forms and of flows. Chapter 6 contains a review of homology.

For details, I highly recommend John Lee's book "Introduction to Smooth manifolds" [**6**].

Crash course on manifolds and differential forms

Here, a good and quick reference is Guillemin-Pollack's book "Differential Topology" [**4**], Chapters 1 (for manifolds) and 4 (for differential forms).

A **manifold** is a (Hausdorff, second countable) topological space M equipped with an equivalence class of **atlases**.

An **atlas** is an open covering $M = \bigcup_i U_i$ and homeomorphisms $\varphi_i\colon U_i \to \Omega_i$ where $\Omega_i \subset \mathbb{R}^n$ is open, such that the transition maps $\varphi_j \circ \varphi_i^{-1}\colon \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ are smooth (that is, are of type C^∞: all partial derivatives of all orders exist and are continuous).

Two atlases $\{\varphi_i\colon U_i \to \Omega_i\}$ and $\{\tilde{\varphi}_j\colon \tilde{U}_j \to \tilde{\Omega}_j\}$ are **equivalent** if their union is an atlas, that is, if $\tilde{\varphi}_j \varphi_i^{-1}$ and $\varphi_i \tilde{\varphi}_j^{-1}$ are smooth for all i, j.

$\varphi_i\colon U_i \to \Omega_i$ is called a **coordinate chart**.

$\varphi_i^{-1}\colon \Omega_i \to U_i$ is called a **parametrization**.

One can write $\varphi_i = (x^1, \dots, x^n)$. $\quad x^j\colon U_i \to \mathbb{R}$ are **coordinates**.

Let $X \subset \mathbb{R}^N$ be any subset and $\Omega \subset \mathbb{R}^n$ open. A continuous map $\varphi\colon X \to \Omega$ is called **smooth** if every point in X is contained in an open subset $V \subset \mathbb{R}^N$ such that there exists a smooth function $\tilde{\varphi}\colon V \to \Omega$ with $\tilde{\varphi}|_{X \cap V} = \varphi$. A continuous map $\psi\colon \Omega \to X$ is called **smooth** if it is smooth as a map to $\mathbb{R}^N$. A **diffeomorphism** $\varphi\colon X \to \Omega$ is a homeomorphism such that both φ and φ^{-1} are smooth.

Theorem. Let $M \subset \mathbb{R}^N$ be a subset that is "locally diffeomorphic to $\mathbb{R}^n$": for every point in M there exists a neighborhood $U \subset M$ and there exists an open subset $\Omega \subset \mathbb{R}^n$ and there exists a diffeomorphism $\varphi\colon U \to \Omega$. Then M is a manifold with atlas $\{\varphi\colon U \to \Omega\}$. (Exercise: the transition maps are automatically smooth.) Such an M is called an **embedded submanifold of $\mathbb{R}^N$.**

Example. $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$. For instance, x, y are coordinates on the upper hemisphere.

A continuous function $f: M \to \mathbb{R}$ is **smooth** if $f \circ \varphi_i^{-1}: \Omega_i \to \mathbb{R}$ is smooth for all i (as a function of n variables).

$C^\infty(M) := \{$ the smooth functions $f: M \to \mathbb{R}$ $\}$.

A (continuous) curve $\gamma: \mathbb{R} \to M$ is **smooth** if $\varphi_i \circ \gamma$ is smooth for all i.

We will define the **tangent space** $T_m M$ = "directions along M at the initial point m".

A smooth curve $\gamma: \mathbb{R} \to M$ with $\gamma(0) = m$ defines "differentiation along the curve", which is the linear functional $C^\infty(M) \to \mathbb{R}$,

$$D_\gamma: f \mapsto \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)).$$

We define an equivalence of such curves by $\gamma \sim \tilde{\gamma}$ if $D_\gamma = D_{\tilde{\gamma}}$.

This means that γ and $\tilde{\gamma}$ have the **same direction** at the point $m = \gamma(0) = \tilde{\gamma}(0)$, that is, they are **tangent** to each other at this point.

Geometric definition of the tangent space:

$$T_m M = \{ \text{ the equivalence classes of curves in } M \text{ through } m.\}$$

Leibnitz property: $D_\gamma(fg) = (D_\gamma f)g(m) + f(m)(D_\gamma g)$.

Definition. A **derivation at** m is a linear functional $D: C^\infty(M) \to \mathbb{R}$ that satisfies the Leibnitz property.

Theorem. The derivations at m form a linear vector space: if D_1, D_2 are derivations and $a, b \in \mathbb{R}$ then $aD_1 + bD_2$ is a derivation.

Theorem. If $x^1, \dots, x^n$ are coordinates near m then every derivation is a linear combination of $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$. (The proof uses Hadamard's lemma: for any $f \in C^\infty(\mathbb{R}^n)$ there exist $f_i \in C^\infty(\mathbb{R}^n)$ such that $f(x) = f(0) + \sum x_i f_i(x)$. Proof of the lemma: $f(x) - f(0) = \int_0^1 \frac{d}{dt} f(tx)dt = \sum x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx)dt$ by the chain rule.)

Corollary. For every derivation D there exists a curve γ such that $D = D_\gamma$. $T_m M$ is a linear vector space (identified with the space of derivations at m). If $x^1, \dots, x^n$ are coordinates near m then $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ is a basis of $T_m M$.

Differential of a function: $df|_m \in T_m^* M = (T_m M)^*$ is given by

$$df|_m(v) = vf,$$

the derivative of f in the direction of $v \in T_m M$.

If $x^1, \dots, x^n$ are coordinates then

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial x^i}{\partial x^j} = \delta_{ij},$$

so $dx^1, \dots, dx^n$ is the basis of $T_m^* M$ that is dual to the basis $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ of $T_m M$.

In coordinates, $df = \sum_i \frac{\partial f}{\partial x^i} dx^i$.

A **differential form of degree 0** is a smooth function.

A **differential form of degree 1**, $\alpha \in \Omega^1(M)$, associates to each $m \in M$ a linear functional $\alpha_m \in T^*_m M$. In coordinates: $\alpha = \sum_i c_i(x)dx^i$. We require that the coefficients $c_i(x)$ be smooth functions of $x = (x^1, \ldots, x^n)$.

A **differential form of degree 2**, $\alpha \in \Omega^2(M)$, associates to each $m \in M$ an alternating (i.e., anti-symmetric) bilinear form $\alpha_m : T_m M \times T_m M \to \mathbb{R}$. In coordinates: $\alpha = \sum\limits_{i,j} c_{ij}(x)dx^i \wedge dx^j$ (where

$$dx^i \wedge dx^j : (u, v) \mapsto \det \begin{bmatrix} u^i & v^i \\ u^j & v^j \end{bmatrix}$$

if $u = \sum u^k \frac{\partial}{\partial x^k}$ and $v = \sum v^k \frac{\partial}{\partial x^k}$).

A **differential form of degree k**:

$$\alpha = \sum_{i_1, \ldots, i_k} c_{i_1 \ldots i_k}(x) dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

(where $dx^{i_1} \wedge \ldots \wedge dx^{i_i}$ is similarly given by a $k \times k$ determinant).

Exterior derivative:

$$d\alpha = \sum_{i_1, \ldots, i_k, j} \frac{\partial c_{i_1 \ldots i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}.$$

α is **closed** if $d\alpha = 0$; α is **exact** if there exists β such that $\alpha = d\beta$.

De Rham cohomology: $H^k_{dR}(M) = \{\text{closed } k\text{-forms}\}/\{\text{exact } k\text{-forms}\}$.

An **oriented manifold** is a manifold equipped with an equivalence class of oriented atlases. (Jacobians of transition maps must have positive determinants.)

Integration: Let M be an oriented manifold of dimension n. For an n-form with support in a coordinate neighborhood U_i: write it as $f(x)dx^1 \wedge \ldots \wedge dx^n$ where $x^1, \ldots, x^n$ are (oriented) coordinates and take the Riemann integral of f on $\mathbb{R}^n$. For an arbitrary compactly supported form α: choose a **partition of unity** $\rho_i : M \to \mathbb{R}$, $\sum \rho_i = 1$, $\operatorname{supp} \rho_i \subset U_i$, and define

$$\int_M \alpha = \sum_i \int (\rho_i \alpha).$$

Pullback: $f : M \to N$ induces $f^* : \Omega^k(N) \to \Omega^k(M)$. This enables us to integrate a k-form over an oriented k-submanifold. Properties: $(f \circ g)^* = g^* \circ f^*$, $f^* d\alpha = d\, f^* \alpha$, $f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta$.

A **manifold with boundary** is defined like a manifold except that the Ωs are open subsets of the upper half space. Its boundary ∂M is well defined and is a manifold of one dimension less.

Stokes's theorem: $\int_M d\alpha = \int_{\partial M} \alpha$.

$\alpha \in \Omega^k(M)$ is closed iff $\int_N \alpha = 0$ whenever N is the boundary of a compact oriented submanifold-with-boundary of M. If α is exact, $\int_N \alpha = 0$ for every compact oriented submanifold $N \subset M$. (α is exact iff $\int_N \alpha = 0$ for every smooth cycle N in M.) If the integral of a closed form on N is nonzero, informally N "wraps around a hole in M".

Theorem: if M is oriented and compact n-manifold then $\alpha \mapsto \int_M \alpha$ induces an isomorphism $H^n_{\mathrm{dR}}(M) \to \mathbb{R}$.

Multiplicative structure: $[\alpha] \cdot [\beta] = [\alpha \wedge \beta]$ is a well defined ring structure on $H^*_{dR}(M)$. $f\colon M \to N$ induces a ring homomorphism $f^*\colon H^*_{dR}(N) \to H^*_{dR}(M)$.

Crash course on flows

Here, good references are Chapter 8 of "Introduction to Differential Topology" by Bröcker and Jänich [**1**], Chapter 5 of "A Comprehensive Introduction to Differential Geometry", Volume I, by Michael Spivak [**8**], and Chapters 8 and 9 of John Lee's "Introduction to Smooth Manifolds" [**6**].

Let M be a manifold.

A **vector field** X on M is a map that associates to each point $m \in M$ a tangent vector in T_mM, denoted $X|_m$ or $X(m)$, that is smooth in the following sense. In local coordinates $x^1, \dots, x^n$, a vector field has the form $X = \sum a^j(x) \frac{\partial}{\partial x^j}$; we require that the functions $x \mapsto a^j(x)$ be smooth.

A **flow** on M is a smooth one parameter group of diffeomorphisms $\psi_t\colon M \to M$. This means that ψ_0 =identity and $\psi_{t+s} = \psi_t \circ \psi_s$ for all t and s in $\mathbb{R}$ (so that $t \mapsto \psi_t$ is a group homomorphism from $\mathbb{R}$ to $\mathrm{Diff}(M)$, the group of diffeomorphisms of M), and that $(t, m) \mapsto \psi_t(m)$ is smooth as a map from $\mathbb{R} \times M$ to M.

Its **trajectories**, (or *flow lines*, or *integral curves*) are the curves $t \mapsto \psi_t(m)$. The manifold M decomposes into a disjoint union of trajectories. Moreover, if $\gamma_1(t)$ and $\gamma_2(t)$ are trajectories that both pass through a point p, then there exists an s such that $\gamma_2(t) = \gamma_1(t+s)$ for all $t \in \mathbb{R}$. Hence, the velocity vectors of γ_1 and γ_2 at p coincide.

Its **velocity field** is the vector field X that is tangent to the trajectories at all points. That is, the velocity vector of the curve $t \mapsto \psi_t(m)$ at time t_0, which is a tangent vector to M at the point $p = \psi_{t_0}(m)$, is the vector $X(p)$. We express this as

$$\frac{d}{dt}\psi_t = X \circ \psi_t.$$

Conversely, any vector field X on M generates a *local flow*. This means the following. Let X be a vector field. Then there exists an open subset $A \subset \mathbb{R} \times M$ containing $\{0\} \times M$ and a smooth map $\psi\colon A \subset \mathbb{R} \times M$ such that the following holds. Write $A = \{(t, x) \mid a_x < t < b_x\}$ and $\psi_t(x) = \psi(t, x)$.

(1) ψ_0 =identity.
(2) $\frac{d}{dt}\psi_t = X \circ \psi_t$.
(3) For each $x \in M$, if $\gamma\colon (a, b) \to M$ satisfies the differential equation $\dot\gamma(t) = X(\gamma(t))$ with initial condition $\gamma(0) = x$, then $(a, b) \subset (a_x, b_x)$ and $\gamma(t) = \psi_t(x)$ for all t.

Moreover, $\psi_{t+s}(x) = \psi_t(\psi_s(x))$ whenever these are defined. Finally, if X is compactly supported, then $A = \mathbb{R} \times M$, so that X generates a (globally defined) flow. References: Chapter 8 of "Introduction to Differential Topology" by Bröcker and Jänich; Chapter

5 of "A Comprehensive Introduction to Differential Geometry", volume I, by Michael Spivak; John Lee's "Introduction to Smooth Manifolds".

A *time dependent vector field* parametrized by the interval $[0,1]$ is a family of vector fields X_t, for $t \in [0,1]$, that is smooth in the following sense. In local coordinates it has the form $X_t = \sum a^j(t,x)\frac{\partial}{\partial x^j}$; we require a^j to be smooth functions of $(t, x^1, \ldots, x^n)$.

An *isotopy* (or *time dependent flow*) of M is a family of diffeomorphisms $\psi_t \colon M \to M$, for $t \in [0,1]$, such that ψ_0 =identity and $(t,m) \mapsto \psi_t(m)$ is smooth as a map from $[0,1] \times M$ to M.

An isotopy ψ_t determines a unique time dependent vector field X_t such that

$$\text{(0.1)} \qquad \overset{\text{velocity field}}{} \quad \frac{d}{dt}\psi_t = X_t \circ \psi_t.$$

That is, the velocity vector of the curve $t \mapsto \psi_t(m)$ at time t, which is a tangent vector to M at the point $p = \psi_t(m)$, is the vector $X_t(p)$.

A time dependent vector field X_t on M determines a vector field $\tilde{X}$ on $[0,1] \times M$ by $\tilde{X}(t,m) = \frac{\partial}{\partial t} \oplus X_t(m)$. In this way one can treat time dependent vector fields and flows through ordinary vector fields and flows.

In particular, a time dependent vector field X_t, $t \in [0,1]$, generates a "local isotopy" $\psi_t(x) = \psi(t,x)$. If X_t is compactly supported then $\psi_t(x)$ is defined for all $(t,x) \in [0,1] \times M$. If $X_t(m) = 0$ for all $t \in [0,1]$ then there exists an open neighborhood U of m such that $\psi_t \colon U \to M$ is defined for all $t \in [0,1]$.

The *Lie derivative* of a k-form α in the direction of a vector field X is

$$L_X\alpha = \left.\frac{d}{dt}\right|_{t=0} \psi_t^*\alpha$$

where ψ_t is the flow generated by X.

We have

$$L_X(\alpha \wedge \beta) = (L_X\alpha) \wedge \beta + \alpha \wedge (L_X\beta)$$

and

$$L_X(d\alpha) = d(L_X\alpha).$$

These follow from $\psi^*(\alpha \wedge \beta) = \psi^*\alpha \wedge \psi^*\beta$ and $\psi^* d\alpha = d\psi^*\alpha$.

Cartan formula:

$$L_X\alpha = \iota_X d\alpha + d\iota_X\alpha$$

where $\iota_X \colon \Omega^k(M) \to \Omega^{k-1}(M)$ is

$$(\iota_X\alpha)(u_1, \ldots, u_{k-1}) = \alpha(X, u_1, \ldots, u_{k-1}).$$

(Outline of proof: it is true for functions. If it is true for α and β then it is true for $\alpha \wedge \beta$ and for $d\alpha$.)

Let α_t be a time dependent k-form and X_t a time dependent vector field that generates an isotopy ψ_t. Then

$$\frac{d}{dt}\psi_t^*\alpha_t = \psi_t^*(\frac{d\alpha_t}{dt} + L_{X_t}\alpha_t).$$

(Outline of proof: if it is true for α and for β then it is true for $\alpha \wedge \beta$ and for $d\alpha$. Hence, it is enough to prove it for functions.)

The left hand side, applied to a time dependent function f_t and evaluated at $m \in M$, is the limit as $t \to t_0$ of the difference quotient

$$\frac{f_t(\psi_t(m)) - f_{t_0}(\psi_{t_0}(m))}{t - t_0}.$$

This difference quotient is equal to

$$\left(\frac{f_t - f_{t_0}}{t - t_0}\right)(\psi_t(m)) + \frac{f_{t_0}(\psi_t(m)) - f_{t_0}(\psi_{t_0}(m))}{t - t_0}.$$

The limit as $t \to t_0$ of the first summand is

$$\left.\frac{df_t}{dt}\right|_{t=t_0}(\psi_{t_0}(m)) = \left(\psi_{t_0}^* \left.\frac{df_t}{dt}\right|_{t=t_0}\right)(m).$$

The limit as $t \to t_0$ of the second summand is the derivative of f_{t_0} along the tangent vector

$$\left.\frac{d}{dt}\right|_{t=t_0} \psi_t(m) = X_{t_0}(\psi_{t_0}(m));$$

this derivative is

$$(X_{t_0}f_{t_0})(\psi_{t_0}(m)) = \left(\psi_{t_0}^*(L_{X_{t_0}}f_{t_0})\right)(m).$$

The section “Review of homology” was moved to Chapter 6; see Page 51.

CHAPTER 1

1. Symplectic structures and volume

A **symplectic form** on a manifold M is a closed non-degenerate two-form, ω, on M. A **symplectic manifold** is a pair (M, ω) where M is a manifold and where ω is a symplectic form.

By *manifold* we mean a smooth manifold in the sense that transition functions are C^∞.

Being nondegenerate means that[1] "every nonzero vector has a friend": for every $x \in M$ and every $u \in T_xM \setminus \{0\}$ there exists $v \in T_xM$ such that $\omega(u,v) \neq 0$. This is a pointwise property. Being closed means that $d\omega = 0$; this is a local property, not a pointwise property.

Example. The standard symplectic form on $\mathbb{R}^{2n}$ is

$$\omega_{\text{std}} = \sum_{i=1}^{n} dx_i \wedge dy_i,$$

where $x_1, y_1, \ldots, x_n, y_n$ are the coordinates on $\mathbb{R}^{2n}$.

We will prove:

Theorem (Darboux's theorem). *Let (M, ω) be a symplectic manifold. Then near each point there exist coordinates $x_1, y_1, \ldots, x_n, y_n$ such that $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$.*

Coordinates $x_1, \ldots, y_n$ in which $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$ are called **symplectic coordinates**, **Darboux coordinates**, or (in the context of classical mechanics) **canonical coordinates**.

On a symplectic manifold (M, ω), the **Liouville volume form**[2] is $\dfrac{\omega^n}{n!}$. It induces on M an *orientation* and a *measure*. In canonical coordinates, it is $dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n$.

On a two dimensional manifold, a symplectic form is the same thing as an area form, i.e., a nonvanishing two-form.

A **symplectomorphism** from a symplectic manifold (M, ω) to a symplectic manifold (M', ω') is a diffeomorphism $\psi \colon M \to M'$ such that $\psi^* \omega' = \omega$.

[1] quoting Sue Tolman

[2] The results of the next section imply that $\omega^n = \underbrace{\omega \wedge \ldots \wedge \omega}_{n \text{ times}}$ is non-vanishing.

We will prove, modulo some “black boxes”:

Theorem (Gromov’s non-squeezing theorem). *If $\lambda < 1$, then there is no symplectic embedding of the unit ball*[3] $B(1) = \{x_1^2 + y_1^2 + \ldots + x_n^2 + y_n^2 < 1\}$ *into the cylinder* $Z(\lambda) = \{x_1^2 + y_1^2 < \lambda^2\}$, *both equipped with the standard symplectic form* ω_{std} *induced from the ambient Euclidean space* $\mathbb{R}^{2n}$.

Remark.

(i) An embedding of $B(1)$ into $Z(\lambda)$ is a diffeomorphism[4] of $B(1)$ with an open[5] subset of $Z(\lambda)$. It is a *symplectic* embedding if also $\psi^*\omega_{\mathrm{std}} = \omega_{\mathrm{std}}$.
(ii) A *volume preserving* embedding from $B(1)$ into $Z(\lambda)$ does exist if $2n > 2$.

Corollary. *There exist volume preserving diffeomorphisms that cannot be uniformly approximated by symplectic diffeomorphisms.*

Exercise 1.1. Prove that if $\lambda < 1$ then there is no *isometric* embedding of $B(1)$ into $Z(\lambda)$ (with respect to the standard Euclidean distance).

This exercise is meant to be easy. In it, you may use the fact that the image of an isometric embedding $\psi\colon B(1) \to \mathbb{R}^{2n}$ is a ball in $\mathbb{R}^{2n}$ of the same radius. Justifying this fact is another interesting exercise.

Exercise. State and prove the case $n = 1$ of Gromov’s non-squeezing theorem. (This is the trivial case of the theorem.)

Example. Let $M = M' = \mathbb{R}^{2n}$ and $\omega = \omega' = \sum_{j=1}^n dx_j \wedge dy_j$. Then

$$(\, x_1, y_1, \ldots, x_n, y_n\,) \mapsto (\lambda x_1, \lambda^{-1} y_1, \ldots, \lambda x_n, \lambda^{-1} y_n\,)$$

is a symplectomorphism that takes the unit ball into the cylinder $\{\, x_1^2 + \ldots + x_n^2 < \lambda^2\,\}$. (Contrast this with Gromov’s nonsqueezing theorem; note that there is no contradiction.)

Exercise 1.2. Let S^2 be the unit sphere in $\mathbb{R}^3$. The tangent space $T_q S^2$ (as a subspace of $\mathbb{R}^3$) is $q^\perp$. The area form ω on S^2 can be written as

$$\omega|_q(u, v) = \langle q, u \times v\rangle$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{R}^3$ and $\times$ is the vector product on $\mathbb{R}^3$.

(i) ω is invariant under rotations of S^2.
 (a) State this; and (b) Prove this;
 without introducing coordinates. Use the fact that, for a rotation $A\colon \mathbb{R}^3 \to \mathbb{R}^3$, we have $\langle Au, Av\rangle = \langle u, v\rangle$ and $Au \times Av = A(u \times v)$ for all $u, v \in \mathbb{R}^3$.
(ii) Show that, in Cartesian coordinates x, y, z,

$$\omega = x dy \wedge dz + y dz \wedge dx + z dx \wedge dy.$$

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[3] Unfortunately, some of my best friends denote by $B(r)$ the ball of *capacity* r, namely, of radius ρ such that $\pi\rho^2 = r$. Although the capacity is a natural measurement of size in the symplectic context, it would be inappropriate for me to publicly state my opinion on this non-standard notation.

[4] A diffeomorphism is a bijection such that both it and its inverse are smooth.

[5] “Embedding” is defined in footnote 1 on page 21. In our situation, we don’t need to assume that $\psi(B(1))$ is open; openness holds a-posteriori.

(iii) Show that outside the north and south poles, in cylindrical coordinates r, θ, z, with $x = r\cos\theta$ and $y = r\sin\theta$,

$$\omega = d\theta \wedge dz.$$

The last part of this exercise is essentially a result of Archimedes, by which the surface area that is enclosed between two lines of latitude is proportional to the vertical distance between the lines of latitude.

2. Symplectic linear algebra

Let V be a vector space and $\Omega\colon V \times V \to \mathbb{R}$ a non-degenerate anti-symmetric bilinear form. Such a pair (V, Ω) is called a **symplectic vector space**.

Remark. A bilinear form is also called a *covariant 2-tensor* (in particular by physicists). An antisymmetric bilinear form is also called a *2-covector*. A nondegenerate antisymmetric bilinear form is also called a **symplectic tensor**.[6]

The **symplectic orthocomplement** of a linear subspace S of V consists of the "vectors that don't have friends in S":

$$S^\Omega := \{u \in V \mid \Omega(u, v) = 0 \text{ for all } v \in S\}.$$

Lemma (Symplectic Gram-Schmidt). *Let V be a vector space and $\Omega\colon V\times V \to \mathbb{R}$ a non-degenerate anti-symmetric bilinear form. Then there exists a basis $e_1, f_1, \dots, e_n, f_n$ such that $\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0$ and $\Omega(e_i, f_j) = \delta_{ij}$ for all i and j.*

Corollary. *The dimension of a symplectic vector space is even.*

Remark.

(i) Such a basis is called a *symplectic basis*.
(ii) The above property of the basis $e_1, \dots, f_n$ is equivalent to

$$\Omega \;=\; e_1^* \wedge f_1^* + \ldots + e_n^* \wedge f_n^*$$

where $e_1^*, f_1^*, \dots, e_n^*, f_n^*$ is the dual basis. (Recall that for $\alpha, \beta \in V^*$ we have $\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha$, i.e, $(\alpha \wedge \beta)(u, v) = \alpha(u)\beta(v) = \beta(u)\alpha(v)$.)
(c) In the basis $e_1, \dots, e_n, f_1, \dots, f_n$, the bilinear form Ω is represented by the block matrix $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$:

$$\Omega(u, v) = u^T \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} v$$

where we identify u and v with their coordinate representations in $\mathbb{R}^{2n}$.

Proof of the lemma. If $V = \{0\}$, there is nothing to show. Otherwise, since Ω is (nondegenerate, hence) non-zero, there exist vectors e_1, f_1 such that $\Omega(e_1, f_1) \neq 0$;

[6] I learned this term from John Lee [**6**, Chapter 22]. This term is not (yet?) common in the literature but I really like it.

by multiplying one of them by a constant we may arrange that $\Omega(e_1, f_1) = 1$. Fix such vectors and let $W = \operatorname{span}\{e_1, f_1\}$.

We claim that $V = W \oplus W^\Omega$ in the sense that the subspaces W and W^Ω span V and their intersection is trivial. Indeed, $W \cap W^\Omega = \{0\}$ follows from the nondegeneracy of Ω on W, and $W + W^\Omega = V$ follows by writing an arbitrary vector v as $\Omega(v, f_1)e_1 + \Omega(e_1, v)f_1 +$ some remainder term and confirming that the remainder term is in W^Ω.

Exercise 1.3. Ω is nondegenerate on W^Ω.

Arguing by induction on the dimension of the vector space, by the induction hypothesis there exists a basis $e_2, f_2, \ldots, e_n, f_n$ for W^Ω such that $\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0$ and $\Omega(e_i, f_j) = \delta_{ij}$ for all $i, j \in \{2, \ldots, n\}$. The basis $e_1, f_1, e_2, f_2, \ldots, e_n, f_n$ of V is then as required. □

Let V be a vector space and Ω an antisymmetric bilinear form on V. Its null-space is the set of vetors that "do not have friends":

$$\operatorname{Null}\Omega := \{u \mid \Omega(u, \cdot) = 0\,.$$

Consider the quotient map

$$\pi \colon V \to \overline{V} := V / \operatorname{Null}\Omega\,.$$

Exercise 1.4.

(i) There exists a unique antisymmetric bilinear form $\overline{\Omega}$ on $\overline{V}$ such that $\Omega = \pi^*\overline{\Omega}$ (i.e. such that $\Omega(u, v) = \overline{\Omega}(\pi u, \pi v)$ for all u, v).

(ii) $\overline{\Omega}$ is nondegenerate.

Lemma. *Let V be a vector space and Ω and antisymmetric bilinear form on V. Then the following are equivalent.*

- *Ω is nondegenerate.*
- $\operatorname{Null}\Omega = \{0\}$.
- *The map $\Omega^\sharp \colon V \to V^*$ given by $v \mapsto \Omega(v, \cdot)$ is a linear isomorphism.*
- $\dim V = 2n$ *is even, and* $\Omega^n \neq 0$.

Various notations for the *contraction* of Ω with v; also called the *interior product* of Ω with v:

$$\Omega(v, \cdot) = \iota_v\Omega = \iota(v)\Omega = v \lrcorner\, \Omega.$$

Proof of the lemma. The equivalence of the first and second property is from the definition of "nondegenerate". Since $\operatorname{Null}\Omega = \ker\Omega^\sharp$, nondegeneracy is equivalent to $\Omega^\sharp$ being one-to-one; by a dimension count this is equivalent to $\Omega^\sharp$ being onto. If Ω is nondegenerate, then with respect to a symplectic basis we have $\Omega^n = (e_1^* \wedge f_1^* + \ldots + e_n^* \wedge f_n^*)^n = n! e_1^* \wedge f_1^* \wedge \cdots \wedge e_n^* \wedge f_n^*$, which is nonzero. Conversely, assume that Ω is degenerate and $\dim V = 2n$ is even. Then by the exercise $\Omega = \pi^*\overline{\Omega}$ for $\pi \colon V \to \overline{V} = V / \operatorname{Null} V$, and $\dim \overline{V} < \dim V$. So $\Omega^n = \pi^*\overline{\Omega}^n$, which is zero because $\overline{\Omega}^n$, being a $(2n)$-covector on a vector space of dimension $< 2n$, must be zero. □

Let (V,Ω) be a symplectic vector space. A linear subspace $S \subset V$ is **symplectic** if $\Omega|_S$ (meaning $\Omega|_{S\times S}$) is nondegenerate. A linear subspace $S \subset V$ is **isotropic** if $\Omega|_S$ is zero.

EXAMPLE. Let $e_1, f_1, \dots, e_n, f_n$ be a symplectic basis. Then

$$\left(\operatorname{span}\{f_1,\dots,f_n\}\right)^\Omega = \operatorname{span}\{e_1,\dots,e_n\}\,;$$
$$\left(\operatorname{span}\{e_1,f_1\}\right)^\Omega = \operatorname{span}\{e_2,f_2,\dots,e_n,f_n\}\,.$$

Exercise 1.5.

(i) S is isotropic if and only if $S^\Omega \supseteq S$.
(ii) S is symplectic if and only if $V = S \oplus S^\Omega$ (namely, the subspaces S and S^Ω span V and their intersection is trivial).
(iii) Moreover, if S is symplectic, then S^Ω is also symplectic.

3. Hermitian structures

The standard Hermitian structure on $\mathbb{C}^n$ is

$$H(u,v) = \sum_{j=1}^{n} \overline{u}_j v_j$$
$$= g(u,v) + i\omega(u,v).$$

Its real part g is the standard inner product on $\mathbb{R}^{2n}$ and its imaginary part ω is the standard symplectic tensor on $\mathbb{R}^{2n}$, when we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$.

A *complex structure* on a real vector space V is an automorphism $J\colon V \to V$ such that $J^2 = -I$. (It defines the multiplication by $i = \sqrt{-1}$.)

On a real vector space V can consider the following structures:

g an inner product,

ω a symplectic tensor,

J a complex structure.

The *compatibility condition* between these structures is

$$g(u,v) = \omega(u,Jv) \qquad \text{for all } u,v.$$

The compatibility condition implies that any two of ω, g, J determine the third.

EXAMPLE. On a real vector space V with a basis $e_1, f_1, \dots, e_n, f_n$, the structures ω, g, J that are given by

$$\omega = \sum e_j^* \wedge f_j^*\,, \qquad g = \sum e_j^* \otimes e_j^* + f_j^* \otimes f_j^*\,, \qquad Je_j = f_j\,,\ Jf_j = -e_j$$

are compatible.

A Hermitian structure on a complex vector space (V,J) is a sesquilinear map $H \times V \times V \to \mathbb{C}$ (sesquilinear means that it is complex linear in the second variable and anti-complex-linear in the first variable) that satisfies $H(v,u) = \overline{H(u,v)}$ for all u and v and that is positive definite (which means that $H(u,u) > 0$ for all $u \neq 0$).

REMARK.

(i) Such an H is also called a Hermitian inner product or, sometimes, a complex inner product.
(ii) Some authors define omit positive-definiteness in their definition of a Hermitian structure (in contrast to Hermitian inner product).
(iii) Some authors require H to be complex linear in the first variable and anti-complex-linear in the second variable.

Given bilinear forms ω and g on a complex vector space (V, J), the sum $\omega + ig$ is a Hermitian structure if and only if ω is a symplectic tensor, g is an inner product, and ω, g, J are compatible: $g(u,v) = \omega(u, Jv)$.

Exercise. Let V be a real vector space with a symplectic tensor ω, an inner product g, and a complex structure J, that are compatible. Then there exists a real linear isomorphism of V with $\mathbb{C}^n$ that takes ω, g, J to the corresponding standard structures on $\mathbb{C}^n$. (Proof: Gram-Schmidt for the Hermitian structure.)

Let V be a real vector space with a symplectic structure ω, an inner product g, and a complex structure J, that are compatible. The automorphism groups of these structures are the symplectic linear group, the complex general linear group, and the orthogonal group:

$$\begin{aligned}
\mathrm{Sp}(V,\omega) &:= \left\{ \begin{matrix} \text{real linear automorphisms } A\colon V \to V \\ \text{such that } \omega(Au,Av) = \omega(u,v) \text{ for all } u,v \end{matrix} \right\}. \\
\mathrm{GL}_{\mathbb{C}}(V) &:= \{\text{complex linear automorphisms } A\colon V \to V\} \\
&= \{\text{real linear automorphisms } A\colon V \to V \text{ such that } AJ = JA \quad \}. \\
O(V) &:= \{\text{orthogonal } A\colon V \to V\} \\
&= \left\{ \begin{matrix} \text{real linear automorphisms } A\colon V \to V \\ \text{such that } g(Au,Av) = g(u,v) \text{ for all } u,v \end{matrix} \right\}.
\end{aligned}$$

The intersection of any two of these three groups is the unitary group for the Hermitian structure $H = g + i\omega$:

$$U(V) := \left\{ \begin{matrix} \text{complex linear automorphisms } A\colon V \to V \\ \text{such that } H(Au,Av) = H(u,v) \text{ for all } u,v \end{matrix} \right\}.$$

For the standard structures on $V = \mathbb{R}^{2n}$, these groups are the standard groups $\mathrm{Sp}(\mathbb{R}^{2n})$, $\mathrm{GL}_{\mathbb{C}}(n)$, $O(2n)$, and $U(n)$.

Remark (Warning on notation). Take $\mathbb{C}^{2n} = \mathbb{R}^{2n} \otimes \mathbb{C}$ with the complex bilinear extension $\omega_{\mathbb{C}}$ of ω. In Lie theory, the intersection of $U(2n)$ with the group $\mathrm{Aut}(\mathbb{C}^{2n}, \omega_{\mathbb{C}})$ of complex linear automorphisms of $\mathbb{C}^{2n}$ that preserve $\omega_{\mathbb{C}}$ is often called "the symplectic group" and denoted $\mathrm{Sp}(2n)$. But this group is not isomorphic to our group $\mathrm{Sp}(\mathbb{R}^{2n})$ of symplectic linear automorphisms of $\mathbb{R}^{2n}$. Both of these groups are real forms of $\mathrm{Aut}(\mathbb{C}^{2n}, \omega_{\mathbb{C}})$, in the sense that the complexifications of their Lie algebras are equal to the Lie algebra of $\mathrm{Aut}(\mathbb{C}^{2n}, \omega_{\mathbb{C}})$. But, for example, the group $\mathrm{Sp}(2n)$ from Lie theory is compact (it is a maximal compact subgroup of $\mathrm{Aut}(\mathbb{C}^{2n}, \omega_{\mathbb{C}})$), and our group $\mathrm{Sp}(\mathbb{R}^{2n})$ is not compact.

The action of the general linear group $\mathrm{GL}(V)$ on V induces

- a linear action on the vector space of bilinear forms $V \times V \to \mathbb{R}$, and
- a linear action on the vector space of linear maps $V \to V$.

Namely, $A \in \mathrm{GL}(V)$ takes the bilinear form $\eta: V \times V \to \mathbb{R}$ to $(A^{-1})^*\eta$, where $((A^{-1})^*\eta)(u,v) := \eta(A^{-1}u, A^{-1}v)$, and takes the linear map $L: V \to V$ to A_*L, where $A_*L := A \circ L \circ A^{-1}$. (We take $(A^{-1})^*\eta$ and not $A^*\eta$ because our convention is that, unless said otherwise, groups actions are left actions and not right actions.)

These actions take symplectic forms to symplectic forms, inner products to inner products, and complex structures to complex structures. For a vector space V with a fixed choice of a symplectic structure ω, an inner product g, and a complex structure J, the corresponding subgroups $\mathrm{Sp}(V,\omega)$, $\mathrm{GL}_{\mathbb{C}}(V)$, and $O(V)$ are the stabilizers in $\mathrm{GL}(V)$ of ω, J, and g.

CHAPTER 2

4. Compatible complex structures on a symplectic vector space

Fix a symplectic vector space (V, ω).

- An inner product g on V is **compatible** with ω if there exists a complex structure J such that $g(u,v) = \omega(u, Jv)$ for all u, v.
- A complex structure J on V is **compatible** with ω if there exists an inner product g such that $g(u,v) = \omega(u, Jv)$ for all u, v.

We denote

$$\mathcal{J}(V,\omega) \coloneqq \{\text{compatible complex structures } J\}, \text{ and}$$
$$\mathcal{G}(V,\omega) \coloneqq \{\text{compatible inner products } g\}.$$

Thus, the compatibility condition $g(u,v) = \omega(u, Jv)$ defines a bijection

$$\mathcal{J}(V,\omega) \to \mathcal{G}(V,\omega).$$

We have the following embeddings into $(2n)^2$ dimensional vector spaces:

$$\mathcal{J}(V,\omega) \subset \{\text{linear maps } V \to V\};$$
$$\mathcal{G}(V,\omega) \subset \{\text{bilinear forms } V \times V \to \mathbb{R}\}.$$

We will show that $\mathcal{J}(V,\omega)$ and $\mathcal{G}(V,\omega)$ are embedded submanifolds of these ambient vector spaces and the bijection $\mathcal{J}(V,\omega) \to \mathcal{G}(V,\omega)$ is a diffeomorphism. Moreover, we will show that $\mathcal{J}(V,\omega)$ and $\mathcal{G}(V,\omega)$ are contractible, and, furthermore, that they are diffeomorphic to a (lower dimensional) vector space.

The same formula $g(u,v) = \omega(u, Jv)$ defines a linear isomorphism $J \mapsto g$ from the ambient vector space of linear maps $V \to V$ to the ambient vector space of bilinear forms $V \times V \to \mathbb{R}$. The bijection $\mathcal{J}(V,\omega) \mapsto \mathcal{G}(V,\omega)$ is the restriction of this linear isomorphism, so it is a diffeomorphism,[1] and in particular a homeomorphism.

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[1] A map from a subset X of a Euclidean space $\mathbb{R}^N$ to a Euclidean space $\mathbb{R}^K$ is *smooth* if each point of X has a neighbourhood in $\mathbb{R}^N$ on which the map extends to a smooth map to $\mathbb{R}^K$; a map from X to a subset Y of $\mathbb{R}^K$ is smooth it if is smooth as a map to $\mathbb{R}^K$.

A composition of smooth maps is smooth. Smooth maps are continuous with respect to the subset topologies.

A *diffeomorphism* from X to Y is a bijection that is smooth and whose inverse is smooth. An *embedding* of X into $\mathbb{R}^K$ is a diffeomorphism of X with a subset of $\mathbb{R}^K$.

A subset X of $\mathbb{R}^N$ is an n dimensional *embedded submanifold* if and only if for each point of X there exists a relative open neighbourhood U in X and an open subset Ω of $\mathbb{R}^n$ and a diffeomorphism $\varphi: U \to \Omega$. If so, the set of such diffeomorphisms is an atlas.

Because $\mathcal{J}(V,\omega)$ and $\mathcal{G}(V,\omega)$ are homeomorphic, to show that they are contractible, it is though to show that one of them is contractible.

Lemma (Crucial lemma, first version.). *Fix a symplectic vector space* (V,ω). *Then there exist a retraction*

$$\pi\colon\{\textit{inner products on } V\}\to\mathcal{G}(V,\omega).$$

Recall that a retraction of a topological space to a subset is a map to the subset that is continuous and that takes each element of the subset to itself.

Note that the set of inner products on V is convex in the vector space of bilinear forms $V\times V\to\mathbb{R}$.

Corollary. $\mathcal{J}(V,\omega)$ *is contractible.*

Proof of the corollary. Fix any $g_0\in\mathcal{G}(V,\omega)$. Define $F_t\colon\mathcal{G}(V,\omega)\to\mathcal{G}(V,\omega)$, for $t\in[0,1]$, by

$$F_t(g)=\pi\left((1-t)g+tg_0\right).$$

Since π is a retraction, $F_0(g)=g$ and $F_1(g)=g_0$ for all $g\in\mathcal{G}(V,\omega)$. So F_t is a homotopy between the identity map and a constant map. Having such a homotopy means that $\mathcal{G}(V,\omega)$ is contractible. So $\mathcal{J}(V,\omega)$, being homeomorphic to $\mathcal{G}(V,\omega)$, is contractible too. □

Consider a real vector space V and a linear operator $B\colon V\to V$. The *transpose* of B with respect to an inner product g is the operator B^T such that $g(u,Bv)=g(B^Tu,v)$ for all u,v. The operator B is *symmetric with respect to an inner product* g if it is self-adjoint: $g(u,Bv)=g(Bu,v)$ for all u,v. This holds if and only if B is diagonalizable and its eigenspaces are g-orthogonal. The operator B is symmetric and *positive definite* with respect to g if, additionally, $g(u,Bu)>0$ for all $u\neq 0$. This holds if and only if B is diagonalizable, its eigenspaces are g-orthogonal, and its eigenvalues are positive.

We now prove the crucial lemma (first version), modulo two missing details. At this point, the proof might not seem very intuitive.

Partial proof of the first version of the crucial lemma.

Start with an inner product g on V.

Define an operator A on V by $g(u,v)=\omega(u,Av)$.

Let A^T be the transpose of A with respect to g.

Then A^TA is positive definite and is symmetric with respect to g. So for all $\alpha\in\mathbb{R}$ we can take the operator $(A^TA)^\alpha$. Namely, A^TA is diagonalizable with positive eigenvalues $\lambda_1,\dots,\lambda_n$, and we define $(A^TA)^\alpha$ to be the operator with the same eigenspaces and with the eigenvalues $\lambda_1^\alpha,\dots,\lambda_n^\alpha$. Then $(A^TA)^\alpha$ is symmetric and positive definite with respect to g, and any operator that commutes with A^TA (which is equivalent to preserving the eigenspaces of A^TA) also commutes with $(A^TA)^\alpha$.

Let $J = A(A^T A)^{-\frac{1}{2}}$. We claim but not yet show that $J \in \mathcal{J}(V,\omega)$; this is the first missing detail.

We define $\pi(g)$ to be the corresponding element of $\mathcal{G}(V,\omega)$. Varying g, we get a map $\pi\colon \{\text{inner products on } V\} \to \mathcal{G}(V,\omega)$. We claim but not yet show that this map is continuous; this is the second missing detail.

If we start with g that is already in $\mathcal{G}(V,\omega)$, then the operator A is the corresponding complex structure J, and we obtain that $\pi(g) = g$. So π is a retraction. □

Furthermore, everything here is $\mathrm{Sp}(V,\omega)$ equivariant:

The linear action of $\mathrm{Sp}(V,\omega)$ on (V,ω) induces

- a linear action on the vector space of bilinear forms $V \times V \to \mathbb{R}$ that preserves the subset $\mathcal{G}(V,\omega)$, and
- a linear action on the vector space of linear maps $V \to V$ that preserves the subset $\mathcal{J}(V,\omega)$.

Namely, $A \in \mathrm{Sp}(V,\omega)$ takes the bilinear form $g\colon V \times V \to \mathbb{R}$ to $(A^{-1})^* g$, where $((A^{-1})^* g)(u,v) \coloneqq g(A^{-1}u, A^{-1}v)$, and takes the linear map $L\colon V \to V$ to $A_* L$, where $A_* L \coloneqq A \circ L \circ A^{-1}$. (We take $(A^{-1})^* g$ and not $A^* g$ because our convention is that, unless said otherwise, groups actions are left actions and not right actions.)

The linear isomorphism $J \mapsto g$ from the space of linear maps $V \to V$ to the space of bilinear forms $V \times V \to \mathbb{R}$ that is defined by the formula $g(u,v) = \omega(u, Jv)$ is $\mathrm{Sp}(V,\omega)$ equivariant. Because the $\mathrm{Sp}(V,\omega)$ actions on these spaces take an inner product to an inner product and a complex structure to a complex structure, it follows that these actions preserves the subsets $\mathcal{J}(V,\omega)$ and $\mathcal{G}(V,\omega)$. Thus, the group $\mathrm{Sp}(V,\omega)$ acts on $\mathcal{J}(V,\omega)$ and on $\mathcal{G}(V,\omega)$, and the diffeomorphism $\mathcal{J}(V,\omega) \to \mathcal{G}(V,\omega)$ is $\mathrm{Sp}(V,\omega)$-equivariant.

The retraction $\pi\colon \{\text{inner products on } V\} \to \mathcal{G}(V,\omega)$ that is constructed in the proof of the crucial lemma is also $\mathrm{Sp}(V,\omega)$-equivariant. This equivariance will be made explicit in our upcoming second version of this crucial lemma.

In contrast, the resulting contraction of $\mathcal{J}(V,\omega)$ is not $\mathrm{Sp}(V,\omega)$-equivariant. It relies on a choice of a Hermitian structure and is equivariant only with respect to the corresponding unitary group.

Because $\mathcal{J}(V,\omega)$ and $\mathcal{G}(V,\omega)$ are diffeomorphic, to show that they are embedded submanifolds of the corresponding ambient vector spaces and furthermore that they are diffeomorphic to a (lower dimensional) vector space, it is enough to show this for one of them. We will show this in a later section.

5. Polar decomposition of the linear symplectic group

Fix a vector space V. We can define B^α, for any $\alpha \in \mathbb{R}$ and any linear operator B in the set

$$\{B\colon V \to V \mid \mathrm{Spec}(B) \subset \mathbb{C} \smallsetminus (-\infty, 0]\},$$

by the Cauchy formula

$$B^\alpha := \frac{1}{2\pi i} \oint_{z\in\Gamma} z^\alpha (zI - B)^{-1} dz,$$

where Γ is a counterclockwise curve in $\mathbb{C} \smallsetminus (\infty, 0]$ that surrounds $\mathrm{Spec}(B)$, and where we take the branch of z^α on $\mathbb{C} \smallsetminus (-\infty, 0]$ in which $x > 0$ implies x^α.

Notes:

- Recall that $\mathrm{Spec}(B) = \{z \mid \nexists\, (zI - B)^{-1}\}$.
- A-priori, the formula defines a complex linear operator on $V \otimes \mathbb{C}$. Moreover, the same formula defines B^α for any $\alpha \in \mathbb{C}$.
- The integral is independent of the choice of such Γ. So we may choose Γ to be symmetric about reflection through the real axis. With this choice and when $\alpha \in \mathbb{R}$, we see that the complex linear operator B^α on $V \otimes \mathbb{C}$ is invariant under complex conjugation (exercise!), so it restricts to a real linear operator on V, which we continue to denote B^α.

Properties:

- If B is diagonalizable with positive eigenvalues $\lambda_1, \ldots, \lambda_n$, then B^α is diagonalizable with eigenvalues $\lambda_1^\alpha, \ldots, \lambda_n^\alpha$ on the same eigenspaces.
- If B commutes with an operator A, so does B^α.
  (If B is diagonalizable, commuting with B is equivalent to preserving the eigenspaces of B.)
- For every operator A, we have $(ABA^{-1})^\alpha = AB^\alpha A^{-1}$. I.E., $B \mapsto B^\alpha$ is equivariant with respect to conjugation.
  (If B is diagonalizable, conjugating B has the same affect as acting on the eigenspaces of B while keeping the eigenvalues.)
- The map $(B, \alpha) \mapsto B^\alpha$ is smooth (because the integrand is smooth in (B, α) and the integral is over a fixed compact curve).
- With respect to any inner product $\langle , \rangle$ on V,
  - If B is symmetric, so is B^α.
  - If B is symmetric and positive definite, so is B^α.

We can similarly define $f(B)$ for any analytic function f that is defined on an open subset of $\mathbb{C}$ that contains $\mathrm{Spec}(B)$. In particular we will be interested in the functions $\exp(B)$ (where the definition agrees with the one using power series) and $\log(B)$ (with respect to a branch of the logarithm function that takes positive real numbers to real numbers).

Criteria for $\mathrm{Sp}(\mathbb{R}^n)$ and for $\mathfrak{sp}(\mathbb{R}^{2n})$.

Consider $\mathbb{R}^{2n}$ with its standard inner product and standard symplectic tensor. I will now resort to coordinates. In fact, the criteria below depend on a choice of inner product but not on a choice of coordinates. For fun, you can try to rewrite them in a coordinate-free way. The standard symplectic tensor is represented by the matrix $\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. With our current conventions, this is the negative of the matrix that represents the complex structures.

We also consider the Lie algebra of the symplectic linear group:

$$\mathfrak{sp}(V,\omega) := \{b: V \to V \mid \omega(bu,v) + \omega(u,bv) = 0 \ \forall u,v\}.$$

Criteria:

$$B \in \mathrm{Sp}(\mathbb{R}^{2n}) \text{ if and only if } B^T = \Omega B^{-1}\Omega^{-1}.$$
$$b \in \mathfrak{sp}(\mathbb{R}^{2n}) \text{ if and only if } b^T = \Omega(-b)\Omega^{-1}.$$

Indeed, $\omega(u,v) = u^T\Omega v$ for all u,v, so $\omega(Bu,Bv) = u^T B^T \Omega B v$ for all u,v, and they are equal if and only if $\Omega = B^T\Omega B$, which is equivalent to the condition above.

Similarly, $b \in \mathfrak{sp}(V,\omega)$ if and only if $u^T b^T \Omega v + u^T \Omega b v = 0$ for all u,v, which is equivalent to $b^T\Omega + \Omega b = 0$, which is equivalent to the condition above.

Exercise 2.1.

(a) If $B \in \mathrm{Sp}(\mathbb{R}^{2n})$, then $B^T \in \mathrm{Sp}(\mathbb{R}^{2n})$. If $b \in \mathfrak{sp}(\mathbb{R}^{2n})$, then $b^T \in \mathfrak{sp}(\mathbb{R}^{2n})$.

(b) For any $\alpha \in \mathbb{R}$, if B is symmetric and positive definite and is in $\mathrm{Sp}(\mathbb{R}^{2n})$, then B^α is symmetric and positive definite and is in $\mathrm{Sp}(\mathbb{R}^{2n})$.

(Use the criteria for $\mathrm{Sp}(\mathbb{R}^{2n})$ and for $\mathfrak{sp}(\mathbb{R}^{2n})$.)

Polar decomposition of the symplectic linear group.

Consider $\mathbb{R}^k$ with the standard inner product. The map

$$O(k) \times \{\text{symmetric positive definite matrices}\} \xrightarrow{\cong} \mathrm{GL}(\mathbb{R}^k)$$

given by

$$C, B \mapsto A := CB$$

is invertible, with inverse $A \mapsto (C,B)$ with

$$B := (A^TA)^{\frac{1}{2}} \quad , \quad C := AB^{-1}.$$

Exercise 2.2. Show that B is symmetric and positive definite and that C is orthogonal.

Since the map $(C,B) \mapsto CB$ and its inverse $A \mapsto (\, A(A^TA)^{-\frac{1}{2}} \,,\, (A^TA)^{\frac{1}{2}} \,)$ are smooth, the polar decomposition map is a diffeomorphism.

Claim. *If $k = 2n$ is even then, in the polar decomposition, A is in $\mathrm{Sp}(\mathbb{R}^{2n})$ if and only if B and C are in $\mathrm{Sp}(\mathbb{R}^{2n})$.*

Exercise 2.3. Prove the claim. (Use the criterion for $\mathrm{Sp}(\mathbb{R}^{2n})$.)

Recall that $O(2n) \cap \mathrm{Sp}(\mathbb{R}^{2n}) = U(n)$. So, by the claim, the polar decomposition of the general linear group restricts to a diffeomorphism

$$U(n) \times \{\text{symmetric positive definite matrices in } \mathrm{Sp}(\mathbb{R}^{2n})\} \xrightarrow{\cong} \mathrm{Sp}(\mathbb{R}^{2n}),$$

given by $(C,B) \mapsto A := C \cdot B$, with inverse given by $A \mapsto (C,B)$ with $B = (A^TA)^{\frac{1}{2}}$ and $C = AB^{-1}$.

It will be convenient for us to consider the polar decomposition of the symplectic linear group to be the analogous diffeomorphism with the factors reversed:

$$\{\text{symmetric positive definite matrices in } \mathrm{Sp}(\mathbb{R}^{2n})\} \times U(n) \xrightarrow{\cong} \mathrm{Sp}(\mathbb{R}^{2n}),$$

$(B, C) \mapsto B \cdot C$. This is again a diffeomorphism, and it is equivariant with respect to the right action of $U(n)$ by right multiplication on the $U(n)$ factor in the domain and on $\mathrm{Sp}(\mathbb{R}^{2n})$ in the target.

We would like to show that the space $\mathcal{J}(V, \omega)$ of compatible complex structures is a manifold, diffeomorphic to a vector space. For simplicity, we take (V, ω) to be $\mathbb{R}^{2n}$ with its standard symplectic structure, J_0 the standard complex structure, and $U(V) = U(n)$.

The polar decomposition implies that inclusion into $\mathrm{Sp}(\mathbb{R}^{2n})$ descends to a homeomorphism

$$\{\text{symmetric positive definite matrices in } \mathrm{Sp}(\mathbb{R}^{2n})\} \to \mathrm{Sp}(\mathbb{R}^{2n})/U(n).$$

We would like to argue that this homeomorphism is in fact a diffeomorphism, and, furthermore, that $\mathrm{Sp}(\mathbb{R}^{2n})/U(n)$ is diffeomorphic to $\mathcal{J}(V, \omega)$. To make sense of this, we need to declare a manifold structure on $\mathrm{Sp}(\mathbb{R}^{2n})/U(n)$ or at the very least some structure that will allow us to make sense of "diffeomorphic".

Here are two approaches.

Fact: $\mathrm{Sp}(\mathbb{R}^{2n})$ (as a subset of the vector space of matrices) is a manifold, and there exists a unique manifold structure on $\mathrm{Sp}(\mathbb{R}^{2n})/U(n)$ such that the quotient map from $\mathrm{Sp}(\mathbb{R}^{2n})$ is a submersion. Accepting this fact, the polar decomposition implies that the set of symmetric positive definite matrices in $\mathrm{Sp}(\mathbb{R}^{2n})$ (as a subset of the vector space of matrices) is also a manifold, and that inclusion into $\mathrm{Sp}(\mathbb{R}^{2n})$ descends to a diffeomorphism

$$\{\text{symmetric positive definite matrices in } \mathrm{Sp}(\mathbb{R}^{2n})\} \to \mathrm{Sp}(\mathbb{R}^{2n})/U(n).$$

Alternatively: In the next section we show that the set of symmetric positive definite matrices in $\mathrm{Sp}(\mathbb{R}^{2n})$ (as a subset of the vector space of matrices) is diffeomorphic to a vector space, hence is a manifold. The polar decomposition then implies that $\mathrm{Sp}(\mathbb{R}^{2n})$ (as a subset of the vector space of matrices) is a manifold, that there exists a unique manifold structure on $\mathrm{Sp}(\mathbb{R}^{2n})/U(n)$ such that the quotient map from $\mathrm{Sp}(\mathbb{R}^{2n})$ is a submersion, and that inclusion into $\mathrm{Sp}(\mathbb{R}^{2n})$ descends to diffeomorphism from $\{$symmetric positive definite matrices in $\mathrm{Sp}(\mathbb{R}^{2n})\}$ to $\mathrm{Sp}(\mathbb{R}^{2n})/U(n)$.

We also have a diffeomorphism

$$\mathrm{Sp}(V, \omega)/U(n) \to \mathcal{J}(V, \omega),$$

which takes the coset of an element A of $\mathrm{Sp}(\mathbb{R}^{2n})$ to the push-forward $A_* J_0$ ($= A J_0 A^{-1}$), where J_0 is the standard complex structure on $\mathbb{R}^{2n}$. In an exercise in the next section your will be checking that this map is well defined and is a bijection. (Hints: the fact that this map is well defined and one-to-one follows from the fact that $U(n)$ is equal to the stabilizer of J_0 in $\mathrm{Sp}(\mathbb{R}^{2n})$. The fact that this map is onto follows from the fact that every complex vector space with a Hermitian inner product has an orthonormal basis, (which is proved by the Gram-Schmidt procedure for Hermitian inner products) and is hence isomorphic to $\mathbb{C}^n = \mathbb{R}^{2n}$ with its standard Hermitian structure.) Please believe me for a moment that this map is not only a bijection but in fact a diffeomorphism. (Also see the next section.) From the previous paragraphs we conclude that $\mathcal{J}(V,\omega)$ (as a subset of $\{\text{linear maps } V \to V\}$) is diffeomorphic to the space of symmetric positive definite matrices in $\mathrm{Sp}(\mathbb{R}^{2n})$, and that these diffeomorphic spaces are manifolds, diffeomorphic to a vecor space—modulo one missing detail: and it remains to show that the set of symmetric positive definite matrices in $\mathrm{Sp}(\mathbb{R}^{2n})$ is diffeomorphic to a vector space.

6. Compatible complex structures, again

Recall:

$$B \in \mathrm{Sp}(\mathbb{R}^{2n}) \text{ if and only if } B^T = \Omega B^{-1}\Omega^{-1}.$$
$$b \in \mathfrak{sp}(\mathbb{R}^{2n}) \text{ if and only if } b^T = \Omega(-b)\Omega^{-1}.$$

The exponential map on matrices defines a diffeomorphism

$$\exp\colon \{\text{symmetric matrices}\} \to \{\text{symmetric positive definite matrices}\};$$

its inverse is the logarithm map on matrices.

(The domain is a vector space; the target is an open subset of that vector space.)

Exercise.

(a) If b is symmetric and in $\mathfrak{sp}(\mathbb{R}^{2n})$, then $B := \exp(b)$ is symmetric and positive definite and in $\mathrm{Sp}(\mathbb{R}^{2n})$.
(b) If B is symmetric and positive definite and in $\mathrm{Sp}(\mathbb{R}^{2n})$, then $b := \log(B)$ is symmetric and in $\mathfrak{sp}(\mathbb{R}^{2n})$.

By the exercise, the diffeomorphism from symmetric matrices to symmetric positive definite matrices restricts to a diffeomorphism

$$\exp\colon \left\{\begin{matrix}\text{symmetric matrices}\\ \text{in } \mathfrak{sp}(\mathbb{R}^{2n})\end{matrix}\right\} \to \left\{\begin{matrix}\text{symmetric positive definite matrices}\\ \text{in } \mathrm{Sp}(\mathbb{R}^{2n})\end{matrix}\right\}.$$

So the set of symmetric positive definite matrices that are in $\mathrm{Sp}(\mathbb{R}^{2n})$, (as a subset of the space of all matrices,) is diffeomorphic to a vector space; in particular, it is a manifold.

Substituting this diffeomorphism into the polar decomposition, we get a diffeomorphism

$$\left\{\begin{matrix}\text{symmetric matrices}\\ \text{in } \mathfrak{sp}(\mathbb{R}^{2n})\end{matrix}\right\} \times U(n) \to \mathrm{Sp}(\mathbb{R}^{2n})$$

given by $(b,C) \mapsto (\exp b)\cdot C$, with inverse $A \mapsto (b,C)$ where $b = \log B$ and $C = B^{-1}A$, where $B = (A^TA)^{\frac{1}{2}}$.

Let (V,ω) be $\mathbb{R}^{2n}$ with its standard symplectic structure and standard inner product; let J_0 be its standard complex structure.

Exercise 2.4.

(a) Prove that the map $A \mapsto A_* J_0$ from $\mathrm{Sp}(V,\omega)$ to $\mathcal{J}(V,\omega)$ descends to a bijection $\mathrm{Sp}(V,\omega)/U(n) \to \mathcal{J}(V,\omega)$.
(b) *Optional:* Prove that $\mathcal{J}(V,\omega)$ is closed in the space of linear maps $V \to V$.
(c) *Optional:* Prove that the map $A \mapsto A_* J_0$ from $\mathrm{Sp}(V,\omega)$ to $\mathcal{J}(V,\omega)$ is proper.

Remark.

(i) In the previous section we sketched why this map is a bijection; here I'd like you to add just a bit of details to that sketch.
(ii) I remind you that (for a continuous map between topological spaces) being proper means that the preimage of every compact set is compact. Note that the properness of the map $\mathrm{Sp}(V,\omega) \to \mathcal{J}(V,\omega)$ implies the properness of the map $\mathrm{Sp}(V,\omega)/U(n) \to \mathcal{J}(V,\omega)$.

Keeping in mind the action of $\mathrm{Sp}(\mathbb{R}^{2n})$ on $\{\text{linear maps } V \to V\}$, for a moment let's discuss smooth Lie group actions on manifolds. Let a Lie group G act on a manifold M, let x be a point of M, and let H be the stabilizer of x. Then H is closed, G/H is a manifold (in the sense that it has a unique manifold structure such that the quotient map $G \to G/H$ is a submersion), and the map $a \mapsto a \cdot x$ from G to M descends to a bijection $G/H \to M$. This bijection is an injective immersion. Moreover, it is a weak embedding = diffeological embedding = diffeomorphism with its image in the diffeological sense. In class we didn't prove these facts nor even define diffeological(=weak) embeddings or submanifolds.[2]

Consequently, $\mathcal{J}(V,\omega)$ is a diffeological (= weakly embedded) submanifold of the space of linear maps $V \to V$. But if a weak embedding is also proper then it is an embedding, so the exercise above implies that $\mathcal{J}(V,\omega)$ is an *embedded* submanifold of the space of linear maps $V \to V$, and the map $\mathrm{Sp}(V,\omega)/U(V) \to \mathcal{J}(V,\omega)$ is a diffeomorphism. Since the domain of this map is diffeomorphic to a vector space (namely to the vector space of symmetric matrices that are in $\mathfrak{sp}(\mathbb{R}^{2n})$), so is $\mathcal{J}(V,\omega)$.

Lemma (Crucial lemma, Version 2). *There exists an $\mathrm{Sp}(V,\omega)$-equivariant smooth strong deformation retraction from the set of inner products on V to $\mathcal{G}(V,\omega)$.*

[2]References for group actions and for diffeological embeddings = weak embeddings:

- Sections 5 ("Submanifolds") and 7 ("Lie groups") of John Lee's "Introduction to smooth manifolds";
- Appendix B ("Proper actions of Lie groups") of my 2002 book with Ginzburg and Guillemin (google our names to find some online version); (and an erratum is posted on my website;)
- My 2011 paper "Smooth Lie group actions are parametrized diffeological subgroups" with Patrick Iglesias-Zemmour.

This means that there exist $\mathrm{Sp}(V,\omega)$-equivariant maps

$$\pi_t\colon \{\text{inner products on } V\} \to \{\text{inner products on } V\}, \quad 0 \le t \le 1,$$

such that

(1) $(t,g) \mapsto \pi_t(g)$ is smooth.
(2) π_0 is the identity map.
(3) $\pi_t(g) = g$ for all $g \in \mathcal{G}(V,\omega)$.
(4) $\operatorname{image} \pi_1 = \mathcal{G}(V,\omega)$.

The proof below is the same as in Version 1, except that it contains more details.

Proof of the crucial lemma. Start with any inner product g. Define $A = A_g$ by $g(u,v) = \omega(u, A_g v)$. Note that $g \mapsto A_g$ is smooth.

Let A^T be the transpose of A with respect to g. We claim that $A^T = -A$. Indeed,

$$g(Au,v) = \omega(Au,Av) = \omega(-Av,Au) = g(-Av,u) = g(u,-Av).$$

So $-A^2$, being equal to $A^T A$, is symmetric and positive definite with respect to g, and so it is contained in the domain of definition of $B \mapsto B^\alpha$ for $\alpha \in \mathbb{R}$. Define $\pi_t(g)$ by

$$\pi_t(g)(u,v) = \omega(u, A(-A^2)^{-\frac{t}{2}} v).$$

Note that $(t,g) \mapsto \pi_t(g)$ is smooth and that π_0 is the identity map. Also note that if $g \in \mathcal{G}(V,\omega)$ then A is the corresponding complex structure and so $-A^2$ is the identity map and $\pi_t(g) = g$.

We claim that $\pi_t(g)$ is an inner product. Indeed, by the definition of $A = A_g$, we have that

$$\pi_t(g)(u,v) = g(u, (-A^2)^{-\frac{t}{2}} v).$$

Because $(-A^2)$ is symmetric and positive definite with respect to g, so is $(-A^2)^{-\frac{t}{2}}$, which means that the right hand side of the above equality defines an inner product.

It remains to show that $\pi_1(g) \in \mathcal{G}(V,\omega)$. We already know that $\pi_1(g)$ is an inner product, so we only need to show that

$$J := A(-A^2)^{-\frac{1}{2}}$$

is a complex structure. Indeed,

$$J^2 = (A(-A^2)^{-\frac{1}{2}})^2 = A^2((-A^2)^{-\frac{1}{2}})^2 = A^2(-A^2)^{-1} = -I$$

where the first equality is from the definition of J and the second is because ($-A^2$, hence) $(-A^2)^{-\frac{1}{2}}$ commutes with A. □

CHAPTER 3

7. Weinstein's proof of Darboux's theorem using Moser's method

THEOREM 7.1 (Darboux's theorem). *Let (M,ω) be a $2n$ dimensional symplectic manifold, and let $m \in M$. Then there exists a diffeomorphism $\varphi\colon U \to \Omega$ from an open neighbourhood U of m in M to an open subset Ω of $\mathbb{R}^{2n}$ such that $\varphi^*\omega_{\text{std}} = \omega$. I.E., writing $\varphi = (x_1, y_1, \ldots, x_n, y_n)$, we have $\omega = dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n$.*

You can find a proof of Darboux's theorem that uses using coordinates and induction on n in Shlomo Sternberg's 1964 book "Lectures on differential geometry" or in Arnold's 1978 "Mathematical methods of classical mechanics". In 1969 Alan Weinstein gave a really nice proof of Darboux's theorem, which also applies to Banach manifolds (induction on the dimension won't work when the dimension is infinite), using a method that Jürgen Moser introduced in 1965. We will give here Weinstein's more general "local normal form", which implies Darboux's theorem. (Exercise: deduce Darboux's theorem from Weinstein's local normal form.)

THEOREM 7.2. *Let M be a manifold and $N \subseteq M$ an embedded submanifold. Let ω_0 and ω_1 be closed two-forms on neighbourhoods of N in M that agree and are non-degenerate at the points of N (i.e., on $TM|_N$). Then there exist open neighbourhoods U_0 and U_1 of N in M and a diffeomorphism $\psi\colon U_0 \to U_1$ such that $\psi(x) = x$ for all $x \in N$ and such that $\psi^*\omega_1 = \omega_0$.*

We begin to prove this theorem.

First, we interpolate:

$$\omega_t := (1-t)\omega_0 + t\omega_1.$$

- ω_t is a smooth family of two-forms, defined near N.
- $\omega_t = \omega_0 = \omega_1$ at the points of N. Consequently,
    - ω_t is non-degenerate along N, hence near N.
    - $\frac{d}{dt}\omega_t = 0$ at the points of N.
- $\frac{d}{dt}\omega_t =: \alpha_t$ is a smooth family of closed two-forms that vanish along N.

In fact, α_t is independent of t.

We pause to state the relative Poincaré lemma.

LEMMA 7.3 (Relative Poincaré lemma). *Let $i\colon N \hookrightarrow M$ be an embedded submanifold. Let α be a closed k-form near N whose pullback to N is zero. Then there exists a $k-1$ form β near N such that $d\beta = \alpha$. If α vanishes along N, then β can be chosen to also vanish along N.*

(Moreover, if α_t is a smooth family of closed k forms near N whose pullback to N is zero, then there exists a smooth family β_t of $k-1$ forms near N such that $d\beta_t = \alpha_t$, and if the α_t vanish along N, then the β_t can be chosen to also vanish along N.)

We defer the proof of the relative Poincaré lemma and return to the proof of Weinstein's theorem.

Moser's method tells us to look for a smooth family of diffeomorphisms ψ_t for $0 \leq t \leq 1$, defined on neighbourhoods of N, such that $\psi_t^*\omega_t = \omega_0$ for all t. The corresponding velocity vector field will be the time-dependent vector field X_t defined near N that is defined by $\frac{d}{dt}\psi_t = X_t \circ \psi_t$. Conversely, a time-dependent vector field X_t, for $0 \leq t \leq 1$, defined near N and vanishing along N, determines a time-dependent flow ψ_t, for $0 \leq t \leq 1$, defined on neighbourhoods of N, such that ψ_0 is the identity map, and such that $\frac{d\psi_t}{dt} = X_t \circ \psi_t$. This is a consequence of the fundamental theorem of ODEs. See the "crash course on flows" in Chapter 0.

Moser's strategy is to find a time-dependent vector field X_t that vanishes along N such that the corresponding time-dependent flow ψ_t will satisfy $\frac{d}{dt}\psi_t^*\omega_t = 0$.

We differentiate the left hand side of this equation:

$$\frac{d}{dt}\left(\psi_t^*\omega_t\right) = \psi_t^*\left(\frac{d}{dt}\omega_t + L_{X_t}\omega_t\right)$$

where X_t is the velocity vector field of ψ_t. (For this equation, see the "crash course on flows" in Chapter 0.)

By the relative Poincaré lemma,

$$\frac{d}{dt}\omega_t = d\beta$$

where β vanishes along N. By "Cartan's magic formula",

$$L_{X_t}\omega_t = d\iota_{X_t}\omega_t + \iota_{X_t}\underbrace{d\omega_t}_{=0}.$$

So $\frac{d}{dt}(\psi_t^*\omega_t) = \psi_t^* d\left(\beta + \iota_{X_t}\omega_t\right)$, and it is enough to require

$$\beta + \iota_{X_t}\omega_t = 0.$$

Given β, we solve for X_t. Explicitly, for m near N,

$$\omega_t^\sharp \colon T_mM \to T_m^*M\ ,\ v \mapsto \iota_v\omega_t$$

is smooth and invertible, and we take $X_t = (\omega_t^\sharp)^{-1}(-\beta)$.

Since $\beta = 0$ along N, also $X_t = 0$ along N. So we can solve the ODE

$$\Psi_0 = \text{Identity},\ \frac{d}{dt}\psi_t = X_t \circ \psi_t$$

near N and obtain a family of diffeomorphisms ψ_t of neighbourhoods of N that fix N. Since $\psi_0^*\omega_0 = \omega_0$ and $\frac{d}{dt}\psi_t^*\omega_t = 0$ (by our choice of X_t), we obtain $\psi_t^*\omega_t = \omega_0$ for all t.

$\psi := \psi_1$ is then a diffeomorphism from a neighbourhood of N to a neighbourhood of N that fixes N and that satisfies $\psi^*\omega_1 = \omega_0$, as required.

In the following exercise you are expected to imitate the above arguments and reproduce a theorem of Jürgen Moser for which he introduced this method [**7**]. Here you will not need the relative Poincaré lemma. You may want to review the material on flows and on homology in Chapter 0. Here are two relevant facts.

- A time-dependent vector field that is compactly supported can be integrated to a time-dependent flow that is defined everywhere on the manifold.
- On a k dimensional compact oriented manifold, integration over the manifold induces an isomorphism from the kth de Rham cohomology to $\mathbb{R}$.

(Yes, you may quote these facts. You may have seen these facts in an introductory course on manifolds; let me know if you would like me to post a reference.)

Exercise 3.1. Let M be a two dimensional closed (i.e., compact) manifold. Let ω_0 and ω_1 be area forms on M that induce the same orientation on M and that have the same total area:

$$\int_M \omega_0 = \int_M \omega_1.$$

Then there exists a diffeomorphism $\psi : M \to M$ such that $\psi^*\omega_1 = \omega_0$.

8. Homotopy property of the push forward

In preparation for proving the relative Poincaré lemma, we recall the following homotopy property of the fibre integration operator:

Let V be a manifold. Let $i_s : V \to [0,1] \times V$ for $0 \le s \le 1$, be $x \mapsto (s,x)$. Let $\pi_V : [0,1] \times V \to V$ be $\pi_V(s,x) = x$. Then for every $\gamma \in \Omega^k([0,1] \times V)$,

$$i_1^*\gamma - i_0^*\gamma = (\pi_V)_* d\gamma + d(\pi_V)_*\gamma,$$

where

$$(\pi_V)_* : \Omega^k([0,1] \times V) \to \Omega^{k-1}(V)$$

is the pushforward = fibre integration operator. That is, $(\pi_V)_*$ is a homotopy operator between i_0^* and i_1^* as morphisms of differential complexes

$$i_0^*, i_1^* : (\Omega^*([0,1] \times V), d) \to (\Omega^*(V), d).$$

We will soon give the details. We now show how to use this homotopy property to obtain the relative Poincaré lemma.

Let N be a submanifold of M and let V be a tubular neighbourhood of N, identified with a disc bundle in νN. Let $r_t : V \to V$ be fibrewise multiplication by $t \in [0,1]$. In particular, $r := r_0 : V \to N$ is the projection map. Consider

$$R : [0,1] \times V \to V \quad (t,v) \mapsto tv.$$

Then for every $\alpha \in \Omega^k(V)$,

$$i_1^* R^*\alpha = \alpha \text{ and } i_0^* R^*\alpha = r^*\alpha.$$

Applying the homotopy property of the pushforward to $\gamma := R^*\alpha$, we obtain

$$\alpha - r^*\alpha = (\pi_V)_* dR^*\alpha + d(\pi_V)_* R^*\alpha.$$

If α is closed, then $dR^*\alpha = R^*d\alpha = 0$. If the pullback of α to N vanishes, then $r^*\alpha = 0$. So for such α we have $\alpha = d((\pi_V)_* R^*\alpha)$, and we can take $\beta = (\pi_V)_* R^*\alpha$. If also α

vanishes at the points of N, then $R^*\alpha$ vanishes at the points of $[0,1]\times N$, which implies that β vanishes along N. (In coordinates, write $\alpha = \sum f_{I,J}(x,y)dx_I \wedge dy_J$ where y_j are coordinates along N and x_i are coordinates "normal" to N; then $R^*\alpha = \sum f_{I,J}(tx,y)dx_I \wedge dy_J$. If $f_{I,J}(0,y) = 0$ then $R^*\alpha = 0$.)

We now give the details for the homotopy property of the pushforward.

Push-forward of differential forms. = fibrewise integration.

Define $\pi_V\colon [0,1]\times V \to V$ by $(t,x)\mapsto x$. Then

$$(\pi_V)_*\colon \Omega^k([0,1]\times V) \to \Omega^{k-1}(V)$$

is integration with respect to the $t\in[0,1]$ variable.

In local coordinates $x_1,\ldots,x_n$ on V, if

$$\gamma = \sum_{i_1,\ldots,i_k} f_{i_1,\ldots,i_k}(x,t)dx_{i_1}\wedge\ldots\wedge dx_{i_k} + \sum_{i_1,\ldots,i_{k-1}} f_{i_1,\ldots,i_{k-1}}(x,t)dt\wedge dx_{i_1}\wedge\ldots\wedge dx_{i_{k-1}},$$

then

$$(\pi_V)_*\gamma = \sum_{i_1,\ldots,i_{k-1}} \left[\int_0^1 f_{i_1,\ldots,i_{k-1}}(x,t)dt\right] dx_{i_1}\wedge\ldots\wedge dx_{i_{k-1}}.$$

- $(\pi_V)_*$ is well defined;
- $(\pi_V)_*$ is $\mathrm{Diff}(V)$-equivariant.

More generally, for every fibre bundle $\pi\colon E\to V$ with d dimensional fiber, we have

$$\pi_*\colon \Omega^k_{cv}(E) \to \Omega^{k-d}(V)$$

where the domain is the set of differential forms on E that are fiberwise compactly supported. ("cv" stands for "compact vertically".) See Bott and Tu, page 61 (with a slightly different sign conventions).

Homotopy property.

Let $i_t\colon V\to[0,1]\times V$ be the map $x\mapsto(t,x)$.

For every $\gamma\in\Omega^k([0,1]\times V)$,

$$i_1^*\gamma - i_0^*\gamma = (\pi_V)_*d\gamma + d(\pi_V)_*\gamma.$$

That is, $(\pi_V)_*$ is a homotopy operator between the morphisms of differential complexes

$$i_0^*\ ,\ i_1^*\ :\ (\Omega^*([0,1]\times V),d) \longrightarrow (\Omega^*(V),d)\,.$$

It follows that i_0^* and i_1^* induce the same map in cohomology.

Proof. Take γ expressed in coordinates as above. Applying the exterior derivative,

$$d\gamma = \sum_{i_1,\dots,i_k,j} \frac{\partial f_{i_1,\dots,i_k}}{\partial x_j}(x,t)dx_j \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k} + \sum_{i_1,\dots,i_k} \frac{\partial f_{i_1,\dots,i_k}}{\partial t}(x,t)dt \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k} + \sum_{i_1,\dots,i_{k-1},j} \frac{\partial f_{i_1,\dots,i_{k-1}}}{\partial x_j}(x,t)dx_j \wedge dt \wedge dx_{i_1} \wedge \dots \wedge dx_{i_{k-1}}.$$

Applying $(\pi_V)_*$,

$$(\pi_V)_* d\gamma = \sum_{i_1,\dots,i_k} \left[\int_0^1 \frac{\partial f_{i_1,\dots,i_k}}{\partial t}(x,t)dt\right] dx_{i_1} \wedge \dots \wedge dx_{i_k} - \sum_{i_1,\dots,i_{k-1},j} \left[\int_0^1 \frac{\partial f_{i_1,\dots,i_{k-1}}}{\partial x_j}(x,t)dt\right] dx_j \wedge dx_{i_1} \wedge \dots \wedge dx_{i_{k-1}}.$$

Taking the exterior derivative of the expression for $(\pi_V)_*$ that we had earlier, and switching the order of the operators $\frac{\partial}{\partial x_j}$ and $\int_0^1$,

$$d(\pi_V)_*\gamma = \sum_{i_1,\dots,i_{k-1},j} \left[\int_0^1 \frac{\partial f_{i_1,\dots,i_{k-1}}}{\partial x_j}(x,t)dt\right] dx_j \wedge dx_{i_1} \wedge \dots \wedge dx_{i_{k-1}}.$$

Adding,

$$\begin{aligned}(\pi_V)_* d\gamma + d(\pi_V)_*\gamma &= \sum_{i_1,\dots,i_k} \left[\int_0^1 \frac{\partial f_{i_1,\dots,i_k}}{\partial t}(x,t)dt\right] dx_{i_1} \wedge \dots \wedge dx_{i_k} \\ &= \int_0^1 \frac{d}{dt}\left[\sum_{i_1,\dots,i_k} f_{i_1,\dots,i_k}(x,t)dx_{i_1} \wedge \dots \wedge dx_{i_k}\right] = \int_0^1 \frac{d}{dt} i_t^*\gamma = i_1^*\gamma - i_0^*\gamma.\end{aligned}$$

□

9. Lagrangian submanifolds and cotangent bundles

A linear subspace S of a $2n$ dimensional symplectic vector space (V,ω) is

symplectic	if $\omega\vert_S$ is non-degenerate;
isotopic	if $\omega\vert_S = 0$, which is equivalent to $S \subseteq S^\omega$;
coisotropic	if $S \supseteq S^\omega$;
Lagrangian	if $S = S^\omega$, which is equivalent to S being isotropic of dimension n.

Here, $\omega|_S$ means $\omega|_{S\times S}$. We already defined *symplectic* and *isotropic*; we are now defining *coisotropic* and *Lagrangian*. The fact that being Lagrangian is equivalent to being isotropic of dimension n follows from the fact that S and S^ω have complementary dimensions, which can be seen by considering the composition

$$\begin{tikzcd} V \arrow[r, "\omega^\sharp", "\cong"'] \arrow[rr, bend right, "\text{kernel}=S^\omega"'] & V^* \arrow[r, two heads, "\text{restrict to } S"] & S^* \end{tikzcd}$$

and writing $\dim V = \dim \text{kernel} + \dim \text{image}$.

A submanifold $i\colon N \hookrightarrow (M,\omega)$ of a symplectic manifold is

symplectic	if $TN \subseteq TM|_N$ is symplectic, i.e., $i^*\omega$ is symplectic;
isotropic	if $TN \subseteq TM|_N$ is isotropic, i.e., $i^*\omega = 0$;
coisotropic	if $TN \subseteq TM|_N$ is coisotropic,
Lagrangian	if $TN \subseteq TM|_N$ is Lagrangian, i.e., $i^*\omega = 0$ and $\dim N = 2n$.

$TM|_N$ is a symplectic vector bundle over N, which means that it is a vector bundle equipped with a fibrewise symplectic tensor (which varies smoothly as a function of the base coordinates). The subbundle TN is said to be "symplectic", "isotropic", "coisotropic", or "Lagrangian" if this property holds fibrewise.

Similar adjectives apply to an immersion i.

Let S be a Lagrangian subspace of a symplectic vector space (V,ω). The above diagram induces a natural isomorphism $V/S \to S^*$:

$$\begin{tikzcd}
V \arrow[r, "\omega^\sharp"] \arrow[rr, bend right, "\text{kernel}=S^\omega=S"'] \arrow[d] & V^* \arrow[r, "\text{restrict to } S"] & S^* \\
V/S \arrow[urr, bend right, "\cong"'] & &
\end{tikzcd}$$

The isomorphism $V/S \to S^*$ is equivariant with respect to the action of linear symplectomorphisms of V that preserve S. It is also "smooth in S" in the sense that it defines a (smooth) isomorphism of vector bundles over the Lagrangian Grassmannian

$$\begin{tikzcd}
Q \arrow[rr] \arrow[dr] & & D \arrow[dl] \\
& \left\{\begin{matrix}\text{Lagrangian}\\ \text{subspaces}\\ S \subset V\end{matrix}\right\}. &
\end{tikzcd}$$

where $Q|_S = V/S$ and $D|_S = S^*$.

(There are several ways to define the manifold structure on the Lagrangian Grassmannian that are equivalent to each other. If you are familiar with the manifold structure on the ordinary Grassmannian of n-planes in V, we note that the Lagrangian Grassmannian is an embedded submanifold.)

This implies that a Lagrangian embedding $i\colon L \hookrightarrow (M,\omega)$ induces an isomorphism

$$TM|_L/TL \xrightarrow{\cong} T^*L$$

from the normal bundle νL of L in M to the cotangent bundle T^*L of L.

The ordinary tubular neighbourhood theorem gives a diffeomorphism

$$\text{neighbourhood of } L \text{ in } M \xrightarrow{\cong} \text{neighbourhood of the zero section in } \nu L.$$

Alan Weinstein used his local normal form to make this a *symplectomorphism* with respect to a natural symplectic structure on T^*L, which we will now describe.

We recall that the cotangent bundle of a manifold N is the disjoint union

$$T^*N = \bigsqcup_{x\in N} T_x^*N.$$

We can write a point in the cotangent bundle as a pair (x,φ) where $x \in N$ and $\varphi \in (T_xN)^*$. The projection map $\psi: T^*N \to N$ is $\pi(x,\varphi) = x$; its differential takes a tangent vector $\zeta \in T_{(x,\varphi)}(T^*N)$ to a tangent vector $\pi_*\zeta \in T_xN$. The **tautological one-form** α on T^*N is

$$\alpha|_{(x,\varphi)}: \zeta \mapsto \varphi(\pi_*\zeta).$$

With respect to local coordinates $q_1,\dots,q_n$ on N, and the corresponding adapted coordinates $q_1,\dots,q_n,p_1,\dots,p_n$ on T^*N, we have

$$\alpha = \sum_i p_i dq_i.$$

(The q_i on the right are the pullbacks to T^*N of the coordinate functions on N.) This formula implies that the exterior derivative $d\alpha$ is symplectic. We define the canonical symplectic form on the cotangent bundle to be $-d\alpha$. (In the literature, the sign convention varies.) In adapted coordinates,

$$\omega = \sum_i dq_i \wedge dp_i.$$

Thus, any adapted coordinates on T^*N are canonical (=symplectic) coordinates.

(More details: given local coordinates $q_1,\dots,q_n$ on an open subset U of N, every covector $\varphi \in T_x^*N$ can be written uniquely as a linear combination of the covectors $dq_1|_x,\dots,dq_n|_x$, and we define the functions $p_1,\dots,p_n$ on $T^*N|_U$ to be the coefficients of this linear combination. The *adapted coordinates* on $T^*N|_U$ are $(q_1,\dots,q_n,p_1,\dots,p_n)$, where we use the same symbols $q_1,\dots,q_n$ to denote the pullbacks to $T^*N|_U$ of the coordinate functions on U. The topology on T^*N is defined to be the smallest topology such that the sets $T^*N|_U$ are open and the adapted coordinates $q_1,\dots,q_n,p_1,\dots,p_n$ are continuous. The manifold structure on T^*N is defined by the atlas whose maps are the functions $(q_1,\dots,q_n,p_1,\dots,p_n): T^*N|_U \to \mathbb{R}^{2n}$.)

CHAPTER 4

Almost complex structures

10. Weinstein's Lagrangian tubular neighbourhood theorem

THEOREM (Weinstein's Lagrangian tubular neighbourhood theorem). *Let L be a Lagrangian submanifold of a symplectic manifold (M, ω). Then there exists a symplectomorphism from a neighbourhood of L in M to a neighbourhood of the zero section in T^*L whose restriction to L is the zero section.*

The proof uses the following lemma:

LEMMA. *There exists a Lagrangian splitting $TM|_L = TL \oplus E$.*

This means that there exists a Lagrangian sub-bundle E of $TM|_L$ that is complementary to TL.

SKETCH OF PROOF OF THE LEMMA. The "crucial lemma" implies that there exists a compatible almost complex structure $J\colon TM \to TM$; the next section contains some details. Take $E = J(TL)$. (Exercise: check that this E is Lagrangian and is complementary to TL.) $\square$

Given an E as in the lemma, the symplectic form ω gives a nondegenerate pairing between TL and E. Using this to identify E with T^*L, we obtain a Lagrangian splitting of symplectic vector bundles over L,

$$TM|_L \cong TL \oplus T^*L,$$

which at each $x \in L$ takes the given symplectic tensor $\omega|_x$ on T_xM to the standard symplectic tensor on $T_xL \oplus T_x^*L$, which is

$$((v_1, \varphi_1), (v_2, \varphi_2)) \mapsto \varphi_2(v_1) - \varphi_1(v_2)$$

for vectors $v_1, v_2 \in T_xL$ and covectors $\varphi_1, \varphi_2 \in T_x^*L$.

For any vector bundle $E' \to L$ we have

$$TE'|_L = TL \oplus E'$$

where on the left hand side we identify the zero section with L and on the right hand side we identify the tangent at 0 to the fibre E'_x with E'_x. Two special cases are

$$T(\nu L)|_L = TL \oplus \nu L = TL \oplus T^*L$$

(with the identification $\nu L = T^*L$ that is induced from ω), and

$$T(T^*L)|_L = TL \oplus T^*L.$$

In the last equation, evaluating the canonical symplectic form ω_{can} on T^*L at the points of the zero section again gives the standard pairing on $T_xL \oplus T_x^*L$.

The ordinary tubular neighbourhood theorem gives a diffeomorphism ψ from a neighbourhood of L in M to a neighbourhood of L in νL whose restriction to L is the identity map, where we identify L with the zero section. The differential of ψ is then an isomorphism of vector bundles

$$d\psi|_L : TM|_L \to T(\nu L)|_L = TL \oplus \nu L.$$

Moreover, in the ordinary tubular neighbourhood theorem we can prescribe the splitting: for any splitting $TM|_L \cong TL \oplus \nu L$, we can choose the diffeomorphism ψ such that $d\psi|_L$ induces this splitting.

In our case, having chosen a Lagrangian splitting $TM|_L \cong TL \oplus T^*L$, we obtain a diffeomorphism ψ from a neighbourhood of L in M to a neighbourhood of L in $\nu L = T^*L$ such that

$$d\psi|_L : TM|_L \to TL \oplus T^*L$$

is the Lagrangian splitting that we fixed. We then have $\psi^*\omega_{\text{can}}|_x = \omega|_x$ on T_xM for all $x \in L$, where ω_{can} is the canonical symplectic form on T^*L.

Weinstein's local normal form then gives a diffeomorphism

$$\tilde{\psi} : \{\text{neighbourhood of } L \text{ in } M\} \to \{\text{neighbourhood of } L \text{ in } M\}$$

such that $\tilde{\psi}^*(\psi^*\omega_{\text{can}}) = \omega$ and $\tilde{\psi}|_L = \mathrm{Id}_L$. Composing, we obtain a diffeomorphism

$$\psi \circ \tilde{\psi} : \{\text{neighbourhood of } L \text{ in } M\} \to \{\text{neighbourhood of } L \text{ in } T^*L\}$$

that satisfies $(\psi \circ \tilde{\psi})^*\omega_{\text{can}} = \omega$. This completes the proof of Weinstein's Lagrangian tubular neighbourhood theorem.

11. Compatible almost complex structures on a symplectic manifold

An **almost complex structure** on a manifold M is a smooth fibrewise complex structure on TM, i.e., a bundle automorphism $J : TM \to TM$ such that $J^2 = -I$.

A **compatible** almost complex structure on a *symplectic* manifold (M, ω) is an almost complex structure J such that $J|_x$ is a compatible complex structure on the symplectic vector space $(T_xM, \omega|_x)$ for each $x \in M$. We denote

$$\mathcal{J}(M,\omega) := \{\text{compatible complex structures on } (M,\omega)\}.$$

LEMMA. *Let (M,ω) be a symplectic manifold. Then $\mathcal{J}(M,\omega)$ is non-empty. Moreover, $\mathcal{J}(M,\omega)$ is contractible.*

PROOF. The "crucial lemma" of Chapter 2 gives, for each $x \in M$, a retraction π_x from the space of inner products on T_xM to the space of compatible inner products on $(T_xM, \omega|_x)$. These fit together into a retraction

$$\pi : \{\text{Riemannian metrics on } M\} \to \{\text{compatible Riemannian metrics on } (M,\omega)\}$$

where as before "compatible" means compatible on each $(T_xM, \omega|_x)$.

Let g_0 be any Riemannian metric on M. Then $\pi(g_0)$ is a compatible Riemannian metric on M, and we get an element of $\mathcal{J}(M,\omega)$ by taking the corresponding almost complex structure. Thus, $\mathcal{J}(M,\omega)$ is non-empty.

Let g_0 be any compatible Riemannian metric on M. Then $g \mapsto \pi((1-t)g + tg_0)$, for $0 \le t \le 1$, defines a homotopy from the identity map to a constant map, on the space of compatible Riemannian metrics on M. Identifying the space of compatible metrics with $\mathcal{J}(M,\omega)$, we obtain a contraction of $\mathcal{J}(M,\omega)$. □

REMARK. In the above lemma, some details of the statement and of the proof are implicit.

Recall that a Riemannian metric is a smooth choice of inner product on each T_xM. In local coordinates, an inner product is represented by a matrix-valued function of the coordinates, and this function is required to be smooth.

The first detail is that $\pi(g_0)$ satisfies the smoothness condition in the definition of a Riemannian metric. This is a consequence of the smoothness of the retraction in the "crucial lemma". (Exercise: fill the details.)

Next, the topology that we take on $\mathcal{J}(M,\omega)$, which was not made explicit in the statement of the lemma, is the C^∞ topology. In this topology, a sequence of almost complex structures converges if all its derivatives of all orders converge uniformly on compact subsets. More precisely, in a local coordinate chart $\varphi = (x_1,\dots,x_k)\colon U \to \Omega$ where $k = 2n = \dim M$, the almost complex structure becomes a smooth map $J_\varphi\colon U \to \mathbb{R}^{k\times k}$ from the open subset U of M to the space of $k\times k$ matrices. For every multi-index $I = (i_1,\dots,i_k) \in \mathbb{Z}^k_{\ge 0}$, let $|I| = i_1 + \ldots + i_k$, and denote $D_I = \frac{\partial^{|I|}}{\partial x_1^{i_1}\cdots\partial x_k^{i_k}}$. For every multi-index I, chart $\varphi\colon U \to \Omega$, compact subset $K \subset U$, and open subset $\mathcal{O} \subset \mathbb{R}^{k\times k}$, consider the set

$$\{J \in \mathcal{J}(M,\omega) \mid D_I J_\varphi|_q \in \mathcal{O} \text{ for all } q \in K\}.$$

These sets form a sub-basis for the C^∞ topology on $\mathcal{J}(M,\omega)$.

Recall that a retraction from a topological space to a subspace is a continuous map from the ambient space to the subspace that restricts to the identity map on the subspace. With the C^∞ topology, the map π satisfies the continuity requirement in the definition of a retraction. We omit the details.

Moreover, the construction in the proof of the "crucial lemma, Version 2" gives a strong deformation retraction from the space of Riemannian metrics on M to the space of compatible Riemannian metrics on M. This means that it gives a continuous family of maps F_t from the space of Riemannian metrics on M to itself such that F_0 is the identity map, F_1 is a retraction to the space of compatible Riemannian metrics, and each F_t restricts to the identity map on the space of compatible Riemannian metrics.

Furthermore, the deformation retraction F_t is *smooth* in the diffeological sense. This means that if we start with a finite dimensional smooth family of Riemannian metrics then we obtain a finite dimensional smooth family of Riemannian metrics. That is, if for each $y = (y_1,\dots,y_r)$ in some open subset V of some Euclidean space $\mathbb{R}^r$ we have a Riemannian metric g_y such that in local coordinates $x_1,\dots,x_{2n}$ the matrix that represents g_y depend smoothly on $(y_1, \dots, y_r, x_1, \dots, x_{2n})$, then the matrix that represents $F_t(g_y)$ depends smoothly on $(t, y_1, \dots, y_r, x_1, \dots, x_{2n})$.

Exercise 4.1 (Flexibility of compatible almost complex structures). Let (M, ω) be a symplectic manifold and $U \subset M$ an open subset. Let J_1 be a compatible almost complex structure on an open subset of M that contains the closure $\overline{U}$ of U. Then there exists a compatible almost complex structure J_2 on (M, ω) that coincides with J_1 on U.

Hint: use a partition of unity.

12. An application to Lagrangian intersections

Here are two important examples of Lagrangian submanifolds.

- Let (M, ω) be a symplectic manifold. A diffeomorphism $f: M \to M$ is a symplectomorphism if and only if[1] its graph $\{(x, f(x)) \mid x \in M\}$ is a Lagrangian submanifold of $(M \times M, (-\omega) \oplus \omega)$.
- Let N be a manifold. A one-form β on N is closed if and only if [2] its graph $\{(x, \beta|_x) \mid x \in N\}$ is a Lagrangian submanifold of T^*N.

Here is an application of Weinstein's Lagrangian tubular neighbourhood theorem.

Theorem 12.1. *Let (M, ω) be a simply connected compact symplectic manifold of dimension $2n \geq 2$. Let $f: M \to M$ be a C^1-small symplectomorphism. Then f has at least two fixed points.*

The statement of the theorem means that there exists some C^1-neighbourhood[3] of the identity map such that every f in this neighbourhood has at least two fixed points.

We now prove the theorem.

By Weinstein's local normal form, we can identify a neighbourhood U of the diagonal $\{(x, x)\}$ in $(M \times M, (-\omega) \oplus \omega)$ with a neighbourhood of the zero section in T^*M, such that the diagonal map $x \mapsto (x, x) \in M \times M$ becomes the zero section $x \mapsto (x, 0) \in T^*M$.

If $f: M \to M$ is a C^1 small diffeomorphism, then the map $x \mapsto (x, f(x))$ from M to $M \times M$ is C^1 close to the diagonal map $x \mapsto (x, x)$. Weinstein's local normal form

[1]This is a consequence that the pullback of $(-\omega) \oplus \omega$ by the map $x \mapsto (x, f(x))$ is $-\omega + f^*\omega$.

[2]This is a consequence of the fact that the pullback of the tautological one-form on T^*N by the map $(x \mapsto \beta|_x): N \to T^*N$ is equal to β.

[3]The C^1 topology can be taken to be the subset topology when we embed the space of diffeomorphisms $f: M \to M$ into the space of continuous maps $TM \to TM$ by $f \mapsto df$ and take the compact-open topology on the ambient space. If we fix a chart $\varphi = (x_1, \dots, x_k): U \to \Omega \subseteq \mathbb{R}^k$ (with $k = 2n$) on M, a compact subset K of U, and an $\epsilon > 0$, the set of diffeomorphisms f that take K into U and whose coordinate representation $(f_1(x_1, \dots, x_k), \dots, f_k(x_1, \dots, x_k))$ satisfies $|f_j(x) - x_j| < \epsilon$ and $|\frac{\partial f_j}{\partial x_i}(x)| < \epsilon$ for all j and i and for all $x \in K$ is a C^1 neighbourhood of the identity map; the finite intersections of such sets form a local base for the C^1 topology at the identity map.

takes this map to a map $M \to T^*M$ that is C^1 close to the zero section. Its image is then the graph of a one-form[4] β.

The fixed points of f exactly correspond to the zeros of β.

If f is a symplectomorphism, then its graph is Lagrangian in $M \times M$, so the graph of β is Lagrangian in T^*M, and so β is closed. If M is simply connected, every closed one-form is exact. Writing $\beta = dh$ for some smooth function $h\colon M \to \mathbb{R}$, we see that β must have at least two zeros, obtained from the minimum and maximum of h on the compact manifold M. So f must have at least two fixed points.

This argument shows a stronger result: on a simply connected symplectic manifold (M, ω), a C^1 small symplectomorphism has at least as many points as the minimum number of critical points that a smooth function on M must have.

[4]The composition of the map $M \to T^*M$ with the projection to M is C^1 close to the identity map, so it is a diffeomorphism of M. Precomposing the map $M \to T^*M$ with the inverse of this diffeomorphism gives a section of T^*M; take β to be this section.

CHAPTER 5

Holomorphic maps

13. Complex manifolds

Definition: a complex atlas is an atlas whose maps take values in $\mathbb{C}^n$ and whose transition maps are holomorphic. Complex atlases are equivalent if their union is a complex atlas. A complex manifold is a manifold equipped with an equivalence class of complex atlases.

Example. The complex projective space $\mathbb{CP}^n$ is a complex manifold, with the charts $\varphi_j: U_j \to \mathbb{C}^n$ on $U_j := \{[z_0 : z_1 : \ldots : z_n] \mid z_j \neq 0\}$ given by $\varphi_j([z_0 : z_1 : \ldots : z_n]) = (\frac{z_0}{z_j}, \ldots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \ldots, \frac{z_n}{z_j})$.

A function $f: \Omega \to \mathbb{C}$ on an open subset Ω of $\mathbb{C}$ is holomorphic if and only if it is C^1 and its differential $df|_x: \mathbb{C} \to \mathbb{C}$ is complex linear for every $x \in \Omega$. Indeed, writing $z = x + iy$ and $f = g + ih$, the function f is holomorphic if and only if its real and imaginary parts g and h are C^1 and satisfy the Cauchy-Riemann equations $\frac{\partial g}{\partial x} = \frac{\partial h}{\partial y}$ and $\frac{\partial g}{\partial y} = -\frac{\partial h}{\partial x}$. These equations can be written as $df \circ i = i \circ df$, which means that the real linear map df is also complex linear.

For a function $f = (f^{(1)}, \ldots, f^{(m)}): \Omega \to \mathbb{C}^m$ on an open subset Ω of $\mathbb{C}^n$, being holomorphic can be defined as holomorphicity of each $f^{(i)}(z_1, \ldots, z_n)$ separately in each z_j. A function $f: \Omega \to \mathbb{C}^m$ is holomorphic if and only if $df|_x: \mathbb{C}^n \to \mathbb{C}^m$ is complex linear for each $x \in \Omega$, i.e., $df \circ j = J \circ df$ where $j: \mathbb{C}^n \to \mathbb{C}^n$ and $J: \mathbb{C}^n \to \mathbb{C}^n$ are the complex structures (multiplications by $\sqrt{-1}$)).

A complex manifold M is naturally almost complex: for any point $x \in M$ and any complex coordinate chart $U \to \Omega$ with $x \in U \subseteq M$ and $\Omega \subseteq \mathbb{C}^n$, the differential of the chart is a real linear isomorphism $T_xM \xrightarrow{\cong} \mathbb{C}^n$. The induced complex structure on T_xM is independent of the chart because the differentials of the transition maps are complex linear.

A map $\psi: M \to N$ between almost complex manifolds is defined to be **holomorphic** if the differential $d\psi|_x: T_xM \to T_{f(x)}N$ is complex linear for every $x \in M$. Often in the literature this property is called "pseudo-holomorphic", but in fact the term "holomorphic" is not ambiguous: by the above discussion, if the almost complex structures on M and N come from complex structures, then ψ is holomorphic in this new sense if and only if it is holomorphic in the usual sense, when written in terms of the coordinates of complex atlases.

On a $2n$ dimensional almost complex manifold (M, J),

- typically there exist many holomorphic curves (i.e., holomorphic maps from Riemann surfaces to M, for example with domain a disc $D^2 \subset \mathbb{C}$);
- if $\dim M > 2$, typically there are no holmorphic functions $M \to \mathbb{C}$.

If near each point there exists a biholomorphic map $U \to \mathbb{C}^n$, then the set of all such maps is a maximal complex atlas. ("Biholomorphic" means that the map is a bijection and it and its inverse are holomorphic.) Thus, on a complex manifold, the almost complex structure determines the complex structure.

An almost complex structure $J: TM \to TM$ is **integrable** if it comes from a complex structure.

An almost complex structure on a two-dimensional manifold is always integrable. In two dimensions, an almost complex structure is equivalent to a conformal structure (inner product up to scaling) together with an orientation. Moreover, in two dimensions, every conformal structure admits "isothermal" coordinates, i.e., coordinates in which the conformal structure is represented by the standard Euclidean metric. Finding such coordinates amounts to solving the so-called Beltrami equation. favourite reference?

More generally, the obstruction to integrability of an almost complex structure J is given by the Nijenhuis tensor $N_J: TM \otimes TM \to TM$, which is characterized by the fact that, for vector fields X and Y,

$$N_J(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X,Y].$$

(Exercise: this is a tensor. I.E. its value at a point of M depends only on the values of the vector fields X and Y at that point.) The Newlander-Nirenberg theorem asserts that J is integrable if and only if $N_J \equiv 0$. See Hörmander's book on complex analysis or—for the real analytic case—the book by Kobayashi-Nomitzu. If the dimension of the manifold is two, then the vanishing of N_J can be seen by noting that (in any dimension) N_J is antisymmetric and $N_J(X, JX) = 0$.

A **Kähler manifold** is a symplectic manifold (M, ω) equipped with a compatible complex structure. (For complex geometers, a Kähler manifold is a complex manifold equipped with a fibrewise Hermitian structure on TM whose imaginary part—which is a nondegenerate two form—is closed. This is the same as our definition; the only difference is that we start with the symplectic structure and they start with the complex structure.) An **almost Kähler** manifold is simply a symplectic manifold equipped with a compatible almost complex structure.

A submanifold N of a complex manifold M is complex if and only if it is almost complex, i.e., $J(TN) = TN$ where J is the almost complex structure on M.

A complex submanifold N of a Kähler manifold is Kähler, and in particular symplectic. (By compatibility, if v is a non-zero vector in TN, then Jv is a "friend" of v in TN.)

We will later show that $\mathbb{CP}^n$ is Kähler; its symplectic form is called the *Fubini-Study form.* It will follow that every smooth projective variety is symplectic.

14. The Fubini-Study form on $\mathbb{CP}^n$.

Recall that the complex projective space $\mathbb{CP}^n$ is defined as the set of one dimensional complex subspaces of $\mathbb{C}^{n+1}$ and can be identified with the following quotients.

$$\mathbb{CP}^n = (\mathbb{C}^{n+1} \smallsetminus \{0\})/\mathbb{C}^\times = S^{2n+1}/S^1.$$

There exists a unique complex manifold structure on $\mathbb{CP}^n$ such that the quotient map $\mathbb{C}^{n+1} \smallsetminus \{0\} \to \mathbb{CP}^n$ is holomorphic. It is given by the charts

$$\mathbb{CP}^n \supset U_j := \{z_j \neq 0\} \xrightarrow{\varphi_j} \mathbb{C}^n \quad , \quad [z_0, z_1, \dots, z_n] \mapsto (w_0, w_1, \dots, w_{j-1}, w_{j+1}, \dots, w_n)$$

where $w_\ell = z_\ell / z_j$. Moreover, the quotient map $\mathbb{C}^{n+1} \smallsetminus \{0\} \to \mathbb{CP}^n$ is a (holomorphic) principal $\mathbb{C}^\times$-bundle: a local trivialization from the preimage of U_j to $U_j \times \mathbb{C}^\times$ is given by $(z_0, z_1, \dots, z_n) \mapsto ([z_0, z_1, \dots, z_n], z_j)$.

Similarly, the quotient map $\pi \colon S^{2n+1} \to \mathbb{CP}^n$ is a principal S^1 bundle: a local trivialization from $\pi^{-1} U_j$ to $U_j \times S^1$ is given by $(\zeta_0, \zeta_1, \dots, \zeta_n) \mapsto ([\zeta_0, \zeta_1, \dots, \zeta_n], \frac{\zeta_j}{|\zeta_j|})$.

There exists a unique symplectic structure ω_{FS} on $\mathbb{CP}^n$ such that $\pi^* \omega_{\mathrm{FS}} = i^* \omega_{\mathbb{C}^n}$, where i and π are the inclusion and quotient maps:

$$\begin{tikzcd} S^{2n+1} \arrow[r, hook, "i"] \arrow[d, "\pi"] & \mathbb{C}^{n+1} \\ \mathbb{CP}^n & \end{tikzcd}$$

On U_0, we can see this directly by considering the commuting diagram

$$\begin{tikzcd} B^{2n}(1) \times S^1 \arrow[r, "\cong"] \arrow[d, "\text{projection}"'] & \pi^{-1} U_0 \arrow[d, "\pi"] \\ B^{2n}(1) \arrow[r, "\cong"] & U_0 \, , \end{tikzcd}$$

where $B^{2n}(1)$ is the open unit sphere in $\mathbb{C}^n$ and where the top horizontal arrow is the S^1-equivariant diffeomorphism

$$(\, \zeta \, , \, e^{i\theta} \,) \ \mapsto \ (\, \sqrt{1 - |\zeta|^2} \, , \, \zeta \,) \cdot e^{i\theta}$$

You can (please do!) check that the pullback of $i^* \omega_{\mathbb{C}^{n+1}}$ under this diffeomorphism is $\omega_{\mathbb{C}^n} \oplus 0$. In particular, the bottom arrow $\zeta \mapsto [\sqrt{1 - |\zeta|^2}, \zeta]$ is a symplectic embedding $B^{2n}(1) \hookrightarrow \mathbb{CP}^n$ whose image is U_0, and its inverse is a Darboux chart on U_0. A similar argument holds on $U_1, \dots, U_n$.

In fact, we obtain $\mathbb{CP}^n = B^{2n}(1) \sqcup \mathbb{CP}^{n-1}$, even symplectically.

Moreover, $\mathbb{CP}^n$ is Kähler (see below). As mentioned earlier, this implies that smooth complex projective varieties are Kähler. (Note: some people say "projective" to mean that there exists a projective embedding. To obtain the two-form one needs to fix a projective embedding.)

Here is another approach for the existence of ω_{FS}.

Reminders about principal bundles:

Let G be a Lie group. A **principal G bundle** is a manifold P and a (right) G action on P and a manifold B and a G-invariant map $\pi\colon P \to B$ such that for every $p \in B$ there exists a neighbourhood U and a diffeomorphism (a "local trivialization")

$$\pi^{-1}U \to U \times G$$

that is G equivariant and whose composition with the projection map $U \times G \to U$ is π.

Theorem. *Let a compact Lie group G act freely on a manifold P. Then there exists a unique manifold structure on $B := P/G$ such that the quotient map $\pi\colon P \to B$ is a submersion. Moreover, $P \to B$ is a principal G bundle.*

The theorem follows from the slice theorem for compact group actions. The conclusion of the theorem remains true if the group G is not compact but the action of G is proper. See [**3**, Appendix B].

In the case of $\mathbb{CP}^n$ we have

$$\pi\colon S^{2n+1} \to \mathbb{CP}^n.$$

The S^1 action is generated by the vector field $\xi = \sum_{j=0}^{n} x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j}$. We have $\ker \pi = \mathbb{R} \cdot \xi$ at each point of S^{2n+1}. The pullback map gives a bijection

$$\{\text{differential forms on } \mathbb{CP}^n\} \xrightarrow[\cong]{\pi^*} \{\text{basic forms on } S^{2n+1}\}$$

where a differential form is **basic** if

- α is S^1 horizontal (i.e., the contraction of α with ξ vanishes: $\iota_\xi \alpha = 0$); and
- α is S^1 invariant (equivalently, the Lie derivative of α along ξ vanishes: $L_\xi \alpha = 0$).

To obtain the Fubini Study form, we need check that $i^*\omega_{\mathbb{C}^{n+1}}$ is basic. S^1-invariance follows from that of $\omega_{\mathbb{C}^{n+1}}$. We now check that $i^*\omega_{\mathbb{C}^{n+1}}$ is horizontal:

$$\begin{aligned}
\iota_\xi i^* \omega_{\mathbb{C}^{n+1}} &= i^* \iota\Big(\sum x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j}\Big) \sum dx_k \wedge dy_k \\
&= -i^*\Big(\sum x_j dx_j + y_j dy_j\Big) \\
&= -i^*\Big(\frac{1}{2} d \underbrace{\sum ({x_j}^2 + {y_j}^2)}_{\equiv 1 \text{ on } S^{2n+1}}\Big) \\
&= 0
\end{aligned}$$

We now check that ω_{FS} is compatible with the complex structure.

Consider the complex sub-bundle

$$W \subset T\mathbb{C}^{n+1}|_{S^{2n+1}}$$

given by

$$\begin{aligned} W_z &= T_z S^{2n+1} \cap i T_z S^{2n+1} \\ &= z^{\perp} \qquad \text{with respect to the complex inner product} \\ &\subset T_z \mathbb{C}^{n+1} = \mathbb{C}^{n+1}. \end{aligned}$$

Since $d\pi_z \colon W_z \to T_{\pi(z)} \mathbb{CP}^n$ is a complex linear isomorphism for all z (by holomorphicity of π), the almost complex structure $J \colon T\mathbb{CP}^n \to T\mathbb{CP}^n$ is induced from $J \colon W \to W$.

Since W is complementary to the (tangents of the) fibres of the projection $\pi \colon S^{2n+1} \to \mathbb{CP}^n$, the two-form ω_{FS} descends from $\omega_{\mathbb{C}^{n+1}}|_W$.

Because J and $\omega_{\mathbb{C}^{n+1}}$ are compatible on the fibres of W, we conclude that the complex structure and ω_{FS} are compatible on $\mathbb{CP}^n$.

15. Differential forms and vector fields on almost complex manifolds

Let $z_1, \ldots, z_m$ be local complex coordinates on a complex manifold. Write $z_j = x_j + i y_j$ and $\overline{z}_j = x_j - i y_j$. Then, at each point q of the domain of the coordinate chart,

$$dz_j = dx_j + i dy_j \quad \text{and} \quad d\overline{z}_j = dx_j - i dy_j$$

are a basis for the complex vector space

$$\begin{aligned} &\{\text{real linear functionals } T_q M \to \mathbb{C}\} \\ &\quad \cong \{\text{complex linear functionals } T_q M \otimes \mathbb{C} \to \mathbb{C}\} \\ &\quad \cong T_q^* M \otimes \mathbb{C}. \end{aligned}$$

The dual basis, to $TM \otimes \mathbb{C}$, is (at each point)

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right).$$

The standard symplectic structure on $\mathbb{C}^n$ can be written as $\frac{i}{2} \sum dz_j \wedge d\overline{z}_j$.

A function f to $\mathbb{C}$ is holomorphic if and only if $\frac{\partial f}{\partial \overline{z}_j} = 0$ for all j; this is just another way of writing the Cauchy-Riemann equations.

We have $df = \partial f + \overline{\partial} f$ where $\partial f = \sum \frac{\partial f}{\partial z_j} dz_j$ and $\overline{\partial} f = \sum \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j$.

A complex formula for Fubini-Study (see Homework 12 of Cannas da Silva's book [**2**]): the pullback of ω_{FS} via the map

$$\mathbb{C}^n \to \mathbb{CP}^n \quad z \mapsto [1, z]$$

equals

$$\frac{i}{2} \partial \overline{\partial} \log(1 + |z|^2).$$

(Warning: Writing $\zeta_j = \frac{z_j}{\sqrt{1+|z|^2}}$, we have $\omega_{\mathrm{FS}} = \sum_j d\frac{x_j}{\sqrt{1+|z|^2}} \wedge d\frac{y_j}{\sqrt{1+|z|^2}}$. But ζ_j are not complex coordinates. So we cannot write this as $\frac{i}{2} \sum_j \partial \zeta_j \wedge \overline{\partial} \zeta_j$.)

On an almost complex manifold, typically there do not exist holomorphic functions from open subsets of M to $\mathbb{C}$. But there exist "approximately holomorphic" functions, and these can be extremely useful, as we learned from Donaldson []. To start, if (M, w, J) is an almost Kähler manifold and $f\colon M \to \mathbb{C}$ is a smooth function such that $|\overline{\partial} f| < |\partial f|$ whenever $f = 0$, then $\{f = 0\}$ is a symplectic submanifold of M. This follows from the exercise below.

Exercise 5.1. Let $A\colon \mathbb{C}^n \to \mathbb{C}$ be a real linear map. Let $A' = \frac{1}{2}(A - iAi)$ and $A'' = \frac{1}{2}(A + iAi)$ be its complex linear and anti-complex-linear parts. Assume that $|A''| < |A'|$. Then $\operatorname{codim}_{\mathbb{R}}(\ker A) = 2$, and $\ker A$ is symplectic.
(Disclaimer: I'm not sure what's the best way to solve this.)

For a C^1 map $f\colon (M_1, J_1) \to (M_2, J_2)$ between almost complex manifolds, we define $\overline{\partial} f = \frac{1}{2}(df + J_2 \circ df \circ J_1)$. The map f is holomorphic if and only if $\overline{\partial} f = 0$.

For a C^1 map $u\colon (\Sigma, j) \to (M, J)$ from a Riemann surface to an almost complex manifold, we often use the notation $\overline{\partial}_J$ to emphasize the dependence on the choice of J. The map u is holomorphic if and only if $\overline{\partial}_J u = 0$. Such a u is called a holomorphic curve, or a pseudo-holomorphic curve, or a J-holomorphic curve.

On an almost complex manifold M, the complexification of J defines a fibrewise complex linear map $J\colon TM \otimes \mathbb{C}$ with $J^2 = -I$. We have a decomposition $TM = (TM)^{1,0} \oplus (TM)^{0,1}$ into the i and $-i$ eigenspaces of J. We have $(TM)^{1,0} = \{v - iJv \mid v \in TM\}$ and $(TM)^{0,1} = \{v + iJv \mid v \in TM\}$. In the integrable case, in terms of complex coordinates $z_1, \dots, z_n$, we have $(TM)^{1,0} = \operatorname{span}\{\frac{\partial}{\partial z_j}\}$ and $(TM)^{0,1} = \operatorname{span}\{\frac{\partial}{\partial \overline{z}_j}\}$.

Dually, we still have the decomposition $T^*M \otimes \mathbb{C} = (T^*M)^{1,0} \oplus (T^*M)^{0,1}$ and, on complex valued differential forms, $\Omega^m = \oplus_{k+\ell=m} \Omega^{k,\ell}$. But the exterior derivative usually does not map $\Omega^{k,\ell}$ into $\Omega^{k+1,\ell} \oplus \Omega^{k,\ell+1}$.

Differential forms on a complex manifold decompose as $\Omega^m(M; \mathbb{C}) = \oplus_{k+\ell=m} \Omega^{k,\ell}(M; \mathbb{C})$ where $\Omega^{k,\ell}$ consists of those differential forms that in local coordinates can be written as $\sum a_{IJ}(z) dz_{i_1} \wedge \cdots \wedge dz_{i_k} \wedge d\overline{z}_{j_i} \wedge \cdots \wedge d\overline{z}_{j_\ell}$. The exterior derivative on differential forms then decomposes as $d = \partial + \overline{\partial}$ where $\partial\colon \Omega^{k,\ell} \to \Omega^{k+1,\ell}$ and $\overline{\partial}\colon \Omega^{k,\ell} \to \Omega^{k,\ell+1}$. The equation $d^2 = 0$ implies that $\partial^2 = 0$, $\overline{\partial}^2 = 0$, and $\partial\overline{\partial} = -\overline{\partial}\partial$.

Vector fields with complex coefficients are smooth sections of $TM \otimes \mathbb{C}$. The space of such vector fields decomposes as $\operatorname{Vect}(M; \mathbb{C}) = \operatorname{Vect}^{1,0}(M) \oplus \operatorname{Vect}^{0,1}(M)$. The ordinary Lie brackets of vector fields extends to a Lie bracket operation on $\operatorname{Vect}(M; \mathbb{C})$. If the almost complex structure J is integrable, then each of the subspaces $\operatorname{Vect}^{1,0}$ and $\operatorname{Vect}^{0,1}$ is closed under the Lie brackets (in other words, the sub-bundles $T^{1,0}M$ and $T^{0,1}M$ are involutive). In complex coordinates, $[a(z)\frac{\partial}{\partial z_j}, b(z)\frac{\partial}{\partial z_k}] = a\frac{\partial b}{\partial z_j}\frac{d}{z_k} - b\frac{\partial a}{\partial z_k}\frac{d}{z_j}$. The converse is true too: J is integrable if and only if $[\operatorname{Vect}^{1,0}, \operatorname{Vect}^{1,0}] \subseteq \operatorname{Vect}^{1,0}$. This inclusion is equivalent to the vanishing of the Nijenhuis tensor.

CHAPTER 6

Homological aspects of symplectic manifolds

16. Review of homology and cohomology

Here, good references are Chapter 18 of John Lee's "Introduction to Smooth Manifolds" [**6**], and Chapters 2.1 and 3.1 of Allen Hatcher's "Algebraic Topology" [**5**].

Homology.

Let M be a manifold. A singular k-chain in M is a formal finite integer combination

$$\sum m_j \, \sigma_j$$

of continuous maps $\sigma_j \colon \Delta^k \to M$ from the standard k-simplex to M. The boundary map ∂ takes singular k-chains to singular $(k-1)$-chains for each k. Cycles are chains with zero boundary. Singular homology is

$$H_k(M) = \{k\text{-cycles}\}/\{k\text{-boundaries}\}.$$

Smooth singular homology is defined similarly with *smooth* maps $\sigma_j \colon \Delta^k \to M$. The natural map from smooth singular homology to $H_k(M)$ is an isomorphism.

Functoriality: $f \colon M \to M'$ induces $f_* \colon H_k(M) \to H_k(M')$.

- $(g \circ f)_* = g_* \circ f_*$;
- if $f = \text{identity}_M$ then $f_* = \text{identity}_{H_k(M)}$;
- if f is a diffeomorphism then f_* is an isomorphism.

If M is an oriented closed connected n-dimensional manifold then $H_n(M) \cong \mathbb{Z}$, generated by the **fundamental class** $[M]$.

Consequently, an oriented closed embedded k-submanifold of M, and more generally a continuous map from an oriented closed manifold to M, represents a k-cycle in M.

If M, M' are oriented closed connected manifolds of the same dimension, the **degree** of a map $\Psi \colon M \to M'$ is the integer $d \in \mathbb{Z}$ such that $\Psi_*[M] = d[M']$. If Ψ is an orientation preserving diffeomorphism then $d = 1$; if Ψ is an orientation reversing diffeomorphism then $d = -1$.

(A finite triangulation of an oriented closed connected k-manifold gives a k-cycle that represents the fundamental class. But the definition of the fundamental class pointer within Hatcher? doesn't rely on a triangulation.)

Cohomology.

A singular k-cochain with coefficients in an abelian group R such as $\mathbb{Z}$ or $\mathbb{R}$ is a homomorphism $\varphi\colon\{\text{singular } k\text{-chains}\} \to R$. The differential = coboundary map

$$\delta\colon\{\text{singular } k\text{-cochains}\} \to \{\text{singular } (k+1)\text{-cochains}\}$$

is $(\delta\varphi)(a) = \varphi(\partial a)$; it takes singular k-cochains to singular $(k+1)$-cochains for each k. Singular cohomology is the subquotient of $\{k\text{-cochains}\}$ given by $H^k(M;R) = \ker\delta/\operatorname{image}\delta$ for each k.

Integration of a differential k-form α over a smooth cycle A gives

$$\int_A \alpha \ \in\ \mathbb{R}.$$

(If $A = \sum_j m_j\sigma_j$ with $\sigma_j\colon\Delta^k \to M$, then $\int_A \alpha := \sum_j m_j \int_{\Delta^k} \sigma_j^*\alpha$.)

For $f\colon M \to M'$, we have $\int_{f_*A} \alpha = \int_A f^*\alpha$.

If $f\colon M \to M'$ is a diffeomorphism between oriented closed connected manifolds and α is a differential form on M' of top degree, then $\int_M f^*\alpha = \pm\int_M \alpha$, depending on whether f preserves or reverses orientation.

- By Stokes's theorem, if α is closed then $A \mapsto \int_A \alpha$ is a cocycle, and if α is exact then $A \mapsto \int_A \alpha$ is a coboundary. Consequently, integration on smooth chains gives a homomorphism from $H^k_{\mathrm{dR}}(M)$ to $H^k(M;\mathbb{R})$.
- If a cochain $\varphi\colon\{k\text{-chains}\} \to \mathbb{R}$ is a cocycle, then it vanishes on boundaries; if it is a coboundary, then it vanishes on chains. Consequently, we get a homomorphism from $H^k(M;\mathbb{R})$ to $\operatorname{Hom}(H_k(M);\mathbb{R})$.
- The inclusion of $\{\mathbb{Z}\text{-cochains}\}$ into $\{\mathbb{R}\text{-cochains}\}$ gives a homomorphism from $H^k(M;\mathbb{Z}) \otimes \mathbb{R}$ to $H^k(M;\mathbb{R})$.

These homomorphisms are isomorphisms:

$$H^k_{\mathrm{dR}}(M) \cong H^k(M;\mathbb{R}) \cong \operatorname{Hom}(H_k(M);\mathbb{R}) \cong H^k(M;\mathbb{Z}) \otimes \mathbb{R}.$$

Integral structure. The integral de Rham cohomology $H^k_{\mathrm{dR}}(M)_{\mathbb{Z}}$ is defined to be the image of the composition $H^k(M;\mathbb{Z}) \xrightarrow{\otimes\mathbb{R}} H^k(M;\mathbb{R}) \xrightarrow{\cong} H^k_{\mathrm{dR}}(M)$. It consists of those classes $[\alpha]$ such that $\int_A \alpha \in \mathbb{Z}$ for every smooth cycle A. For an oriented closed connected manifold M, we have an isomorphism

$$H^n_{\mathrm{dR}}(M)_{\mathbb{Z}} \cong \mathbb{Z} \quad \text{via} \quad [\alpha] \mapsto \int_M \alpha.$$

Ring structure. If the abelian group R has a ring structure, as it does when R is $\mathbb{Z}$ or $\mathbb{R}$, then we get the cup product operation $a, b \mapsto a \cup b$ on cohomology with R coefficients. In de Rham cohomology,

$$[\alpha] \cup [\beta] = [\alpha \wedge \beta]$$

for α, β closed differential forms.

17. Cohomology of a symplectic manifold

Let (M, ω) be a $2n$ dimensional compact symplectic manifold. Since ω^n is a volume form and M is compact, $[\omega]^n = [\omega^n] \neq 0$.

Here are some corollaries.

- Let (M, ω) be a $2n$ dimensional compact symplectic manifold. Then there does not exist a symplectic embedding of (M, ω) into any $\mathbb{R}^{2N}$.
- The only even dimensional sphere S^{2n} that admits a symplectic form is S^2.
- The product $S^2 \times S^4$ does not admit a symplectic form.

Here are some examples and some more corollaries, following a preparatory exercise.

Exercise. Show that the integral over $\mathbb{CP}^1$ of the Fubini-Study form ω_{FS} is π. Show that the integral over S^2 of the standard area form is 4π. Find a diffeomorphism $\mathbb{CP}^1 \to S^2$ that pulls back the standard area form to 4 times the Fubini–Study form.

Example. The nonzero homology groups of $\mathbb{CP}^n$ are

$$H_{2k}(\mathbb{CP}^n) = \mathbb{Z}[\mathbb{CP}^k] \quad \text{for} \quad k \in \{0, \ldots, n\},$$

where $\mathbb{CP}^k \hookrightarrow \mathbb{CP}^n$ is induced from $\mathbb{C}^{k+1} \hookrightarrow \mathbb{C}^{n+1}$. The nonzero integral de Rham cohomology groups of $\mathbb{CP}^n$ are

$$H^{2k}_{\mathrm{dR}}(\mathbb{CP}^n)_{\mathbb{Z}} = [\omega^k] \quad \text{for} \quad k \in \{0, \ldots, n\},$$

where $\omega = \frac{1}{\pi}\omega_{\mathrm{FS}}$.

Corollary: If (M, ω) is a symplectic manifold that admits a symplectic embedding into $(\mathbb{CP}^n, \omega_{\mathrm{FS}})$, then $\frac{1}{\pi}[\omega] \in H^2_{\mathrm{dR}}(M)_{\mathbb{Z}}$.

Example. The nonzero homology groups of $S^2 \times S^2$ are

$$\begin{aligned}
H_0(S^2 \times S^2) &= \mathbb{Z}[\text{point}], \\
H_2(S^2 \times S^2) &= \mathbb{Z}[S^2 \times \text{point}] \oplus \mathbb{Z}[\text{point} \times S^2], \\
H_4(S^2 \times S^2) &= \mathbb{Z}[S^2 \times S^2].
\end{aligned}$$

The nonzero integral de Rham cohomology groups of $S^2 \times S^2$ are

$$\begin{aligned}
H^0_{\mathrm{dR}}(S^2 \times S^2)_{\mathbb{Z}} &= \mathbb{Z}[1], \\
H^2_{\mathrm{dR}}(S^2 \times S^2)_{\mathbb{Z}} &= \mathbb{Z}[\pi_1^*\omega] \oplus \mathbb{Z}[\pi_2^*\omega], \\
H^4_{\mathrm{dR}}(S^2 \times S^2)_{\mathbb{Z}} &= \mathbb{Z}[\,\pi_1^*\omega \wedge \pi_2^*\omega\,],
\end{aligned}$$

where π_1, π_2 are the projections to the two components of $S^2 \times S^2$, and where ω is $\frac{1}{4\pi}$ times the standard area form on S^2.

$$\begin{tikzcd}
 & S^2 \times S^2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
S^2 & & S^2.
\end{tikzcd}$$

Corollary: If $a \geq b > 0$ and a/b is irrational, then there does not exist a symplectic embedding of $(S^2 \times S^2, \omega_{a,b})$ into $(\mathbb{CP}^n, r\omega_{\mathrm{FS}})$ for any $r > 0$.

In solving the following exercises you may rely on the above examples.

Exercise 6.1. Prove that if n is even then there does not exist an orientation reversing diffeomorphism of $\mathbb{CP}^n$.

Exercise 6.2. On the product $S^2 \times S^2$, consider the split symplectic structure $\omega_{a,b} = a\omega \oplus b\omega := a\pi_1^*\omega + b\pi_2^*\omega$, where ω is the rotation-invariant area form on S^2 with total areq equal to 1. Prove: if $(S^2 \times S^2, \omega_{a,b})$ is symplectomorphic to $(S^2 \times S^2, \omega_{a',b'})$ and $a \geq b > 0$ and $a' \geq b' > 0$, then $a = a'$ and $b = b'$.

Exercise 6.3. Assuming Gromov's nonsqueezing theorem, show that, if $D^2(a) \times D^2(b)$ is symplectomorphic to $D^2(a') \times D^2(b')$ and $a \geq b > 0$ and $a' \geq b' > 0$, then $a = a'$ and $b = b'$. Here, $D^2(r)$ is the disc of radius r, and products of discs are equipped with the standard symplectic structure $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$.

Any compact Kähler manifold of complex dimension n and real dimension $2n$ has the following *hard Lefschetz property*: reference? (Griffiths-Harris?)

Hard Lefschetz property: For all $k \in \{0, \dots, n\}$, the map

$$a \mapsto a \cup [\omega]^{n-k}$$

is an isomorphism

$$H^k_{\mathrm{dR}}(M) \xrightarrow{\cong} H^{2n-k}_{\mathrm{dR}}(M).$$

Some compact symplectic manifolds—for example, the Kodaira–Thurston manifold—do not have the hard Lefschetz property, hence are not Kähler in the sense that they do not admit compatible complex structures. (This is in contrast to the fact every symplectic manifold admits a compatible *almost* complex structure.) Students: I might tell you about the Kodaira-Thurston manifold in class later in the semester. I will give this a higher priority if you request it.

We do not know a simply connected compact symplectic manifold with a circle action with finitely many fixed points that does not satisfy the hard Lefchetz property.

18. Indecomposable homology classes

A homology class in $H_2(M)$ is **spherical** if it is in the image of the Hurewicz homomorphism $\pi_2(M) \to H_2(M)$, that is, if it is represented by a map $\varphi: S^2 \to M$.

A symplectic manifold (M, ω) is **symplectically aspherical** if $[\omega]$ vanishes on spherical homology classes; equivalently, if for any smooth map $\varphi: S^2 \to M$ we have $\int_{S^2} \varphi^*\omega = 0$.

Example: If $\pi_2(M) = 0$, then (M, ω) is aspherical. In particular, any closed surface of genus ≥ 1 equipped with an area form is an aspherical symplectic manifold.

More examples of aspherical symplectic manifolds: the torus $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ with the standard symplectic form induced from $\mathbb{R}^{2n}$; the cotangent bundle T^*S^2 with the canonical symplectic form (which is exact); more generally, any exact symplectic manifold.

Example: Robert Gompf, "On symplectically aspherical symplectic manifolds with nontrivial π_2", Math. Res. Letters **5** (1998), 599–603. Gompf's example is a symplectic branched covering of the product $\Sigma_{g_1} \times \Sigma_{g_2}$ of two surfaces of positive genera[1] g_1, g_2, branched over a codimension 2 symplectic submanifold that is obtained from the union $(\Sigma_{g_1} \times \{\text{point}\}) \cup (\{\text{point}\} \times \Sigma_{g_2})$ by "smoothing" the double point according to the model $\{z_1 z_2 = 0\} \mapsto \{z_1 z_2 = \epsilon\}$.

(The notion of a *symplectic branched covering*, of a symplectic manifold along a codimension 2 symplectic submanifold, was introduced by Gromov in his thesis/"partial differential relations"? Transversely to the submanifold, it looks like the map $z \mapsto z^k$ on $\mathbb{C}$.)

Let $A \in H_2(M)$ be a spherical homology class with $\omega(A) > 0$. Then A is **indecomposable** if there is no decomposition $A = A' + A''$ such that A' and A'' are spherical and $\omega(A')$ and $\omega(A'')$ are positive.

(Warning: this definition of "indecomposable" is not uniform in the literature.) (Note: if $A = A' + A''$ such that A' and A'' are spherical and $\omega(A')$ and $\omega(A'')$ are positive, then A is spherical and $\omega(A)$ is positive.)

Example: if the subgroup $\omega(\pi_2(M))$ of $\mathbb{R}$ is discrete and is generated by $\omega(A)$, then A is indecomposable.

Examples below will use the following remark.

Remark. For any $u: S^2 \to M_1 \times M_2$ we can write $u = (u_1, u_2)$ where $u_1: S^2 \to M_1$ and $u_2: S^2 \to M_2$. We can deform u by deforming each u_1 and u_2 such that u_1 becomes constant on the upper hemisphere and u_2 becomes constant on the lower hemisphere. On the level of cohomology, we obtain $[u] = [u_1] \times [\text{point}] + [\text{point}] \times [u_2]$.

Non-example: Identify $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. Let $u: \mathbb{CP}^1 \to \mathbb{CP}^1 \times \mathbb{CP}^1$ be the diagonal map, $u(z) = (z, z)$. Then $[u] = [\mathbb{CP}^1 \times \{\text{point}\}] + [\{\text{point}\} \times \mathbb{CP}^1]$. So $[u]$ is not indecomposable in $(\mathbb{CP}^2 \times \mathbb{CP}^2, \omega_{\text{FS}} \oplus \omega_{\text{FS}})$.

Example: Let $M = \mathbb{CP}^1 \times V$ and $\omega = \omega_{\text{FS}} \oplus \omega_V$ where (V, ω_V) is symplectically aspherical. Then $A := [\mathbb{CP}^1 \times \{\text{point}\}]$ is indecomposable.

The proof is deferred to Chapter 9 because we ran out of time.

[1] 'genera' is the plural of 'genus'

CHAPTER 7

Outline of proof of Gromov non-squeezing

19. Passing to a compact target

Recall Gromov's non-squeezing theorem: Let $\lambda > 0$. Assume that there exists a symplectic embedding of the open unit ball $B^{2n}(1)$ into the radius λ cylinder $B^2(\lambda) \times \mathbb{R}^{2n-2}$, where the ball is given by ${x_1}^2 + {y_1}^2 + \ldots + {x_n}^2 + {y_n}^2 < 1$ the cylinder is given by ${x_1}^2 + {y_1}^2 < \lambda^2$, and the symplectic form is $\sum dx_j \wedge dy_j$. Then $1 \leq \lambda$.

Note: "Embedding" means a diffeomorphism with a subset of the target. When the domain and target are open subsets of $\mathbb{R}^m$ for the same m, an embedding—or, more generally, an injective immersion—is necessarily a diffeomorphism with an *open* subset of the target. Indeed, because the map is injective, as a map to its image it has an inverse; by the inverse function theorem, the map is a local diffeomorphism, so its image is open and its inverse is smooth.

(More generally—by Brouwer's 1912 "invariance of domain" theorem, an injective continuous map from an open subset U of $\mathbb{R}^m$ into $\mathbb{R}^m$ is necessarily a homeomorphism with an open subset of $\mathbb{R}^m$. This is a consequence of Brouwer's fixed point theorem (that a continuous function from the closed ball to itself must have a fixed point). Terrence Tao has a nice blog post on this from 2011.)

Back to Gromov non-squeezing: it is enough to prove the following result.

> If there exists $\epsilon > 0$ such that $B^{2n}(r + \epsilon)$ symplectically embeds in $B^2(R) \times \mathbb{R}^{2n-2}$, then $r \leq R$.

Indeed, assume that this result is true. Assume that $B^{2n}(1)$ embeds in $B^2(R) \times \mathbb{R}^{2n-2}$. Let $\epsilon > 0$. The result with $r = 1 - \epsilon$ implies $1 - \epsilon \leq \lambda$. Since $\epsilon > 0$ can be chosen arbitrarily small, $\lambda \geq 1$.

We turn to prove this result. Let

$$\varphi \colon B^{2n}(r + \epsilon) \to B^2(R) \times \mathbb{R}^{2n-2}$$

be a symplectic embedding. The image of the composition

$$\overline{B^{2n}(r + \frac{\epsilon}{2})} \xrightarrow{\varphi} B^2(R) \times \mathbb{R}^{2n-2} \xrightarrow{\text{projection}} \mathbb{R}^{2n-2}$$

is compact, hence bounded; choose λ that is greater than its diameter. Then composing with the projection map $\mathbb{R}^{2n-2} \to \mathbb{R}^{2n-2}/\lambda\mathbb{Z}^{2n-2}$, we still get a symplectic embedding

$$B^{2n}(r + \frac{\epsilon}{2}) \longrightarrow B^2(R) \times \mathbb{R}^{2n-2}/\lambda\mathbb{Z}^{2n-2}.$$

Further composing with the image of the symplectic embedding of $B^2(R)$ (as the complement of a single point) in $(\mathbb{CP}^1, R^2\omega_{\mathrm{FS}})$,[1] We obtain a symplectic embedding

$$\psi\colon B^{2n}(r+\frac{\epsilon}{2}) \longrightarrow (M,\omega)$$

where

$$M = \mathbb{CP}^1 \times \mathbb{R}^{2n-2}/\lambda\mathbb{Z}^{2n-2} \quad \text{and} \quad \omega = R^2\omega_{\mathrm{FS}} \oplus \omega_{\mathrm{std}},$$

where ω_{std} denotes the symplectic form on $\mathbb{R}^{2n-2}/\lambda\mathbb{Z}^{2n-2}$ that is induced from the standard symplectic form on $\mathbb{R}^{2n-2}$.

20. Using J-holomorphic spheres

"Flexibility of almost complex structures" implies that there exists a compatible almost complex structure $J \in \mathcal{J}(M,\omega)$ whose restriction to the open subset $\psi(B^{2n}(r))$ is $\psi_* J_{\mathrm{std}}$, where J_{std} is the almost complex structure on $B^{2n}(r)$ that is induced from the ambient space $\mathbb{R}^{2n} = \mathbb{C}^n$.

A (very) "Big Lemma":

> *For every point p in M there exists a J-holomorphic sphere*
>
> $$\varphi\colon \mathbb{CP}^1 \to M$$
>
> *in the homology class $[\varphi] = [\mathbb{CP}^1 \times \{point\}]$ that passes through p.*

Example. Let $J_0 := J_{\mathbb{CP}^1} \oplus J_{\mathrm{std}}$ be the split almost complex structure on $M = \mathbb{CP}^1 \times \mathbb{R}^{2n}/\lambda\mathbb{Z}^{2n}$, where $J_{\mathbb{CP}^1}$ is induced from the standard complex structure on $\mathbb{CP}^1$ and J_{std} is induced from the standard complex structure on $\mathbb{R}^{2n} = \mathbb{C}^n$. Then, for any point $p = (z_0, v_0)$ in $\mathbb{CP}^1 \times \mathbb{R}^{2n}/\lambda\mathbb{Z}^{2n}$, the map $z \mapsto (z, v_0)$ is a J_0-holomorphic sphere in M in the homology class $[\mathbb{CP}^1 \times \{\text{point}\}]$ that passes through p.

The "Big Lemma" guarantees that, for our choice of J, there exists a J-holomorphic sphere $\varphi\colon \mathbb{CP}^1 \to M$ in the homology class $[\mathbb{CP}^1 \times \{\text{point}\}]$ that passes through the point $\pi(0)$. Fix such a φ. Then the map

$$f := \psi^{-1} \circ \varphi$$

is well defined on the set $N := \{z \in \mathbb{CP}^1 \mid \varphi(z) \in \psi(B^{2n}(r))\}$ and takes values in $B^{2n}(r)$. By its definition, this map fits into the following (pullback) diagram.

$$\begin{tikzcd} N \arrow[r, "f"] \arrow[d, hook, "\text{inclusion}"'] & B^{2n}(r) \arrow[d, "\psi"] \\ \mathbb{CP}^1 \arrow[r, "\varphi"] & M \end{tikzcd}$$

Exercise. The map $f := N \to B^{2n}(r)$ is a proper holomorphic curve that passes through the origin.

[1]The existence of such an embedding is a consequence of the fact that $B^{2m}(1)$ symplectically embeds as an open dense set in $(\mathbb{CP}^m, \omega_{\mathrm{FS}})$.

Here, we take the standard complex structures, on N as an open subset of $\mathbb{CP}^1$ and on $B^{2n}(r)$ as an open subset of $\mathbb{C}^n$. "Proper" means that preimages of compact sets in $B^{2n}(r)$ are compact in N.

By a consequence of the so-called Wirtinger's inequality that we will prove later, the symplectic area of a proper holomorphic curve in $B^{2n}(r)$ that passes through the origin is $\geq \pi r^2$. Thus,

$$\int_N f^*\omega_{\mathrm{std}} \geq \pi r^2.$$

Since $\varphi\colon \mathbb{CP}^1 \to M$ is J-holomorphic, and since J is compatible with ω, the pullback $\varphi^*\omega$ is compatible with the given complex structure on $\mathbb{CP}^1$, hence is an area form compatible with the complex orientation, at each point where $d\varphi \neq 0$ (and is zero at each point where $d\varphi = 0$). This further implies that restricting to a subset of $\mathbb{CP}^1$ does not increase the symplectic area:

$$\int_N \varphi^*\omega \leq \int_{\mathbb{CP}^1} \varphi^*\omega.$$

We now complete the argument.

$$\begin{aligned}
\pi r^2 &\leq \int_N f^*\omega_{\mathrm{std}} && \text{by the aforementioned consequence of Wirtinger's inequality} \\
&= \int_N \varphi^*\omega && \text{because } \varphi = \psi \circ f \text{ and } \psi^*\omega = \omega_{\mathrm{std}} \\
&\leq \int_{\mathbb{CP}^1} \varphi^*\omega && \text{because } \varphi \text{ is holomorphic and } N \subset \mathbb{CP}^1 \\
&= \int_{\mathbb{CP}^1 \times \{\text{point}\}} \omega && \text{because } [\varphi] = [\mathbb{CP}^1 \times \{\text{point}\}] \text{ in } H_2(M) \\
&= R^2\pi && \text{because } \omega = \omega_{\mathrm{FS}} \oplus \omega'.
\end{aligned}$$

The "Big Lemma" does not rely on our particular choice of J. When allowing J to be arbitrary, the "big lemma" can be rephrased as follows. Let $A = [\mathbb{CP}^1 \times \{\text{point}\}] \in H_2(M)$. Consider the Universal moduli space,

$$\mathcal{M}_A := \left\{ (f, J) \mid J \in \mathcal{J}(M, \omega);\ f\colon \mathbb{CP}^1 \to M \text{ is } J\text{-holomorphic ; } [f] = A \right\}.$$

The projection map

$$\pi\colon \mathcal{M}_A \to \mathcal{J}(M, \omega) \quad , \quad (f, J) \mapsto J$$

and the evaluation map

$$\mathrm{ev}\colon \mathcal{M}_A \times \mathbb{CP}^1 \to M \quad , \quad ((f, J), z) \mapsto f(z)$$

fit together into a map

$$\mathcal{M}_A \times \mathbb{CP}^1 \xrightarrow{\pi \times \mathrm{ev}} \mathcal{J}(M, \omega) \times M \quad , \quad ((f, J), z) \mapsto (J, f(z)).$$

The "Big Lemma" then asserts:

The map $\pi \times \mathrm{ev}$ *is onto.*

Here, it is enough to assume that $M = \mathbb{CP}^1 \times M'$ and $\omega = \omega_{\text{FS}} \oplus \omega'$ where (M', ω') is compact and symplectically aspherical.

21. Proper maps (and an exercise)

Digression to point-set topology – proper maps:

Here is an exercise that I'd like you to hand in:

Exercise 7.1. Let $g: A \to B$ be a continuous proper map from a topological space to a metrizable topological space. Prove that g is a closed map.

Parts of this exercise occur as parts of the not-for-credit exercises below.

Recall that a continuous map $g: A \to B$ between topological spaces is **proper** if the preimage of every compact subset of B is a compact subset of A.

EXERCISE.
- The composition of proper maps is proper.
- The inclusion map of a closed subset is proper.
- Thus, the restriction of a proper map to a closed subset of its domain is proper.
- The level sets of a proper map are compact.
- Given a continuous map $g: A \to B$, if g is proper and B' is a subset of B, then the restriction of g to the preimage of B' is proper as a map to B'. (Warning: it might not be proper as a map to B.)
- Given a continuous map $g: A \to B$, if g is a closed map (i.e., images of closed subsets of A are closed subsets of B) and its level sets are compact, then g is a proper map.
- Given a continuous map $g: A \to B$, if B is Hausdorff (in particular, every compact subset of B is closed) and A is compact, then g is proper.

A Hausdorff space B is **compactly generated**[2] if, for every subset C of B, the set C is closed if and only if its intersection with every compact subset of B is compact.[3]

EXERCISE.
- Any metrizable space is compactly generated.
- Any locally compact Hausdorff space is compactly generated.

EXERCISE. Let $g: A \to B$ be a continuous map between topological spaces. Assume that the target space B is Hausdorff and compactly generated.

- If g is proper, then the image of g is closed.

Consequently,

[2]The name "compactly generated" was introduced in Steenrod's 1967 paper "A convenient category of topological spaces". Steenrod's definition of "compactly generated" included the assumption that the space is Hausdorff.

[3]The "only if" direction is automatic; the "if" direction is meaningful.

Let K be a compact subset of B. For any subset of K, if the subset is closed in B then it is closed in K, and if it is closed in K then it is compact. Because we are assuming that B is Hausdorff, this last condition implies that the subset is closed in B. So a Hausdorff space B is compactly generated if and only if, for every subset C of B, the set C is closed if and only if its intersection with every compact subset K of B is closed (closed in B or closed in K; it doesn't matter which one).

- If g is proper, then g is a closed map.
- If g is proper and its image is dense, then g is onto.
- If g is a proper bijection, then g is a homeomorphism with its image.

For any compact symplectic manifold (M,ω), the space $\mathcal{J}(M,\omega)$ of compatible almost complex structures, equipped with the C^∞ topology, is metrizable, hence compactly generated. The map $\pi \times \mathrm{ev}: \mathcal{M}_A \times \mathbb{CP}^1 \to \mathcal{J}(M,\omega) \times M$ is not proper, but it descends to a proper map once we mod out by reparametrizations.

Reparametrization:

Given a J-holomorphic sphere $f: \mathbb{CP}^1 \to M$ and a holomorphic map $h: \mathbb{CP}^1 \to \mathbb{CP}^1$ of degree one, the composition $f \circ h: \mathbb{CP}^1 \to M$ is a J-holomorphic sphere in the same homology class: $[f] = [f \circ h]$ in $H_2(M)$.

We write

$$\mathbb{CP}^1 = \mathbb{C} \sqcup \{\infty\}.$$

Fact:

$$\begin{aligned}
\{ \text{ holomorphic maps } \mathbb{CP}^1 \to \mathbb{CP}^1 \text{ of degree one } \} \\
&= \{ \text{ Möbius transformations of } \mathbb{CP}^1 \ \} \\
&= \left\{ z \mapsto \frac{az+b}{cz+d} \;\middle|\; a,b,c,d \in \mathbb{C} \text{ and } ad - bc \neq 0 \right\} \\
&\cong \ \mathrm{GL}_2(\mathbb{C})/\mathbb{C}^\times \ \cong \ \mathrm{SL}_2(\mathbb{C})/\mathbb{C}^\times \\
&=: \ \mathrm{PSL}_2(\mathbb{C}).
\end{aligned}$$

$h \in \mathrm{PSL}_2(\mathbb{C})$ acts on $\mathcal{M}_A \times \mathbb{CP}^1$ by

$$((f,J),z) \mapsto ((f \circ h^{-1}, J), h(z)).$$

This action preserves $\pi \times \mathrm{ev}$.

Gromov's compactness theorem (which we are not stating here) implies that the map

$$\overline{\pi}: \mathcal{M}_A / \mathrm{PSL}_2(\mathbb{C}) \to \mathcal{J}(M,\omega)$$

that is induced by $\pi: \mathcal{M}_A \to \mathcal{J}(M,\omega)$ is proper.

Exercise. It follows that the map

$$\overline{\pi \times \mathrm{ev}}: (\mathcal{M}_A \times \mathbb{CP}^1)/\mathrm{PSL}_2(\mathbb{C}) \to \mathcal{J}(M,\omega) \times M$$

that is induced from $\pi \times \mathrm{ev}$ is also proper.

Gromov's compactness theorem, for which I recommend the book by Hummel, is another big result that we are not discussing in detail. I am not even including its statement here; I am only noting its consequence, that $\overline{\pi}$ is proper.

Here, I am referring to our particular example with $M = \mathbb{CP}^1 \times \mathbb{R}^{2n}/\lambda\mathbb{Z}^{2n}$ and $A = [\mathbb{CP}^1 \times \{\text{point}\}]$. But for the properness of $\overline{\pi}$ we only need to assume that the symplectic manifold (M,ω) is compact and the homology class A is indecomposable.

Because the map $\overline{\pi \times \mathrm{ev}}$ is proper, the image of the map $\pi \times \mathrm{ev}$ is closed. To conclude the "Big Lemma", it remains to show that this image is dense. This again is a big result; we will later say something about it but we will not prove it in any detail.

CHAPTER 8

Digression: flat connections modulo gauge; momentum maps

I'm not assigning any exercises-for-credit in this chapter.

22. Flat connections modulo gauge (placeholder)

The first two-hour lecture for this week was replaced by a one-hour guest lecture by Lisa Jeffrey, about flat connections on 2-manifolds and representations of the fundamental group of orientable 2-manifolds.

Notes from Lisa's lecture are linked from the course website (thanks, Jesse Frohlich!).

23. An empty section (placeholder)

24. Followup, and momentum maps (placeholder)

Lisa suggested to Yael to give following "punchline" (due to Atiyah and Bott, 1982), which Lisa did not state in her guest lecture, but for which she gave all the ingredients:

The gauge group action on the vector space of all connections is Hamiltonian.

The momentum map[1] *sends a connection to its curvature.*

Yael then gave a followup to Lisa's lecture, which included the definition of "Hamiltonian action" and "momentum map" (for finite dimensional symplectic manifolds). The notes are linked from the course website (thanks, Jesse Frohlich!).

Lie groups and Lie algebras

(See Jesse's notes.)

Hamiltonian group actions; momentum maps; symplectic reduction.

(See Jesse's notes.)

[1] Lisa calls it "moment map". Yael converted a few years ago and now calls it "momentum map". Both words are used in the literature.

CHAPTER 9

25. Area measurements (placeholder, and an exercise)

For now, please see Caleb's or Jesse's notes.

Here is an exercise that I'd like you to hand in:

Exercise 9.1. Let (M, ω, J) be an almost Kähler manifold. Let $\varphi: \mathbb{CP}^1 \to M$ be a J-holomrophic sphere that is not constant. Then the homology class $[\varphi]$ is non-zero in $H_2(M)$.

26. Consequence of Wirtinger's inequality (placeholder)

For now, please see Caleb's or Jesse's notes.

27. Odds and ends for Gromov non-squeezing

More on indecomposable homology classes

At the end of Chapter 6 we gave an example and didn't have time to give its proof. In a moment we recall this example and give its proof.

Remark. For any map $u: S^2 \to M_1 \times M_2$, we can write $u = (u_1, u_2)$ where u_1 and u_2 are respectively maps from S^2 to M_1 and M_2. Deform each of u_1 and u_2 such that u_1 becomes constant on the upper hemisphere and u_2 becomes constant on the lower hemisphere, we obtain a new map, representing the same class in $\pi_2(M)$ as the original map u, from which we obtain the equality $[u] = [u_1 \times c_2] + [c_1 \times u_2]$ in $\pi_2(M)$, hence in $H_2(M)$, where c_1 and c_2 are the constant maps that take values in the basepoints of M_1 and of M_2, respectively.

Here is the example from the end of Chapter 6:

Let $M = \mathbb{CP}^1 \times V$ and $\omega = \omega_{\text{FS}} \oplus \omega_V$ where (V, ω_V) is symplectically aspherical. Then $A := [\mathbb{CP}^1 \times \{\text{point}\}]$ is indecomposable.

Proof. Seeking a contradiction, suppose that $A = A' + A''$ where A' and A'' are spherical classes and $\omega(A') > 0$ and $\omega(A'') > 0$. Applying the above remark to representatives of A' and of A'', and noting that $H_2(\mathbb{CP}^1) = \mathbb{Z}[\mathbb{CP}^1]$, we obtain

$$A' = a' \, [\mathbb{CP}^1 \times \text{point}] + [\text{point} \times B']$$
$$\text{and} \quad A'' = a'' \, [\mathbb{CP}^1 \times \text{point}] + [\text{point} \times B'']$$

with a' and a'' integers and with B', B'' spherical classes in $H_2(V)$.

Because $\omega = \omega_{\mathrm{FS}} \oplus \omega_V$, we have $\omega(A') = a'\omega_{\mathrm{FS}}(\mathbb{CP}^1) + \omega_V(B')$, and similarly for A''.

Necessarily, $a' + a'' = 1$ (for example, we can see this by evaluating $\omega_{\mathrm{FS}} \oplus 0$). Because a' and a'' are integers with sum 1, without loss of generality we may assume that $a' \geq 1$ and $a'' \leq 0$.

Then B'' is a spherical class in $H_2(V)$, and $\omega_V(B'') = \omega(A'') - a''\omega_{\mathrm{FS}}(\mathbb{CP}^1)$ is positive. ($\omega(A'')$ is positive by assumption; $\omega_{\mathrm{FS}}(= \pi)$ is also positive; and $-a''$ is non-negative.) This contradicts the assumption that (V, ω_V) is symplectically aspherical. □

More on Möbius transformations

Here are some details that we didn't give in class but that I would like to include in these notes.

In Chapter 7 we claimed that the holomorphic maps $\mathbb{CP}^1 \to \mathbb{CP}^1$ of degree one are exactly the Möbius transformations. Let's recall why this is true.

Note that every Möbius transformations is a holomorphic map of degree one and that the Möbius transformations form a group.

Recall that a regular point is a point where the differential is onto and that a regular level set is a level set in which all the points are regular. Recall that (when the domain and target have the same dimension) near each regular point the map is a local diffeomorphism and that the degree of the map is equal to the number of points in a regular level set at which this local diffeomorphism preserves orientation minus the number of points in the regular level set at which this local diffeomorphism reverses orientation. (When the map is proper and the target is connected, the resulting number is independent of the choice of the regular level set.)

Let $h: \mathbb{CP}^1 \to \mathbb{CP}^1$ be a holomorphic map. By (a baby case of) the Weierstrass preparation theorem, near each point there are complex local coordinates in which we can write the map as $h(z) = z^k$ for some non-negative integer k. (Thus, h is a branched covering, ramified over the points where $h'(z) = 0$.) Such a point contributes the non-negative number k to the degree (when the degree is calculated by counting points in a nearby regular level set). Thus, if there is such a point with $k \geq 2$, then the degree is ≥ 2. If there is such a point with $k = 0$, then by analytic continuation the map h is constant, and so the degree of h is zero. Thus, if the degree of h is one, then at each such point we have $k = 1$, and so h is a biholomorphism (an invertible map such that it and its inverse are holomorphic).

After composing with a Möbius transformation we may assume that $h(\infty) = \infty$; because h is a diffeomorphism, it then restricts to a biholomorphism $\mathbb{C} \to \mathbb{C}$. Further composing with a linear map, we may assume that $h(0) = 0$ and $h'(0) = 1$. So we can write $h(z) = z + \sum_{k=2}^{\infty} c_k z^k$. Since h is a biholomorphism near ∞, the map $z \mapsto 1/h(1/z)$ has a removable singularity at $z = 0$; this further implies that $c_k = 0$ for all $k \geq 2$.

More on the outline of Gromov's non-squeezing

We recalled in class the outline of Gromov's non-squeezing theorem. See Chapter 7.

We said a few more words on Gromov's compactness theorem. We still didn't state this theorem. But we gave an example that illustrates "bubbling" in the case that the homology class A is *not* indecomposable. Recall that in Chapter 7 we used (without proving) the "Big Lemma", which says that the map

$$\pi \times \mathrm{ev}: \mathcal{M}_A \times \mathbb{CP}^1 \to \mathcal{J}(M,\omega) \times M \quad , \quad ((f,J),z) \mapsto (J, f(z)),$$

is onto. Here we assume that $M = \mathbb{CP}^1 \times V$, that $\omega = \omega_{\mathrm{FS}} \oplus \omega_V$ where (V,ω_V) is compact and symplectically aspherical, and that $A = [\mathbb{CP}^1 \times \{\text{point}\}]$.

Recall that Gromov's compactness theorem (which we didn't prove either) implies that the induced map

$$\overline{\pi \times \mathrm{ev}}: (\mathcal{M}_A \times \mathbb{CP}^1)/\,\mathrm{PSL}_2(\mathbb{C}) \to \mathcal{J}(M,\omega) \times M$$

is proper. So to prove that the map $\pi \times \mathrm{ev}$ is onto it is enough to prove that its image is dense.

The denseness of this image is again a big result that we are not going to prove here. But we will now explain how we should think of this result (even though the actual proof in the literature is more technical than this). We would like to think of the map $\overline{\pi \times \mathrm{ev}}$ as a proper smooth map between manifolds.[1] By Sard's theorem[2], the set of regular values of a smooth map (has a complement of measure zero, hence) is dense. By degree theory[3], because the map is proper and smooth and its target space is connected, if there is a regular value whose level set contains exactly one point, then the preimage of every regular value contains an odd number of points, hence is non-empty. Together with Sard's theorem, we then deduce that the image of the map is dense; because the image is closed, the map is onto.

We would like to apply this idea[4] to the map $\overline{\pi \times \mathrm{ev}}$. Let $J_0 = J_{\mathbb{CP}^1} \oplus J_V$ be a split almost complex structure on $M = \mathbb{CP}^1 \times V$, where $J_{\mathbb{CP}^1}$ is the standard almost complex structure on $\mathbb{CP}^1$ and J_V is a compatible almost complex structure on (V,ω_V). We claim that any J_0-holomorphic sphere $u: \mathbb{CP}^1 \to M = \mathbb{CP}^1 \times V$ in the homology class $[\mathbb{CP}^1 \times \{\text{point}\}]$ must have the form $u(z) = (h(z), q_0)$ for some Möbius transformation $h: \mathbb{CP}^1 \to \mathbb{CP}^1$ and for some point q_0 in V. This follows from Exercise 9.1 and from the fact that a holomorphic map $\mathbb{CP}^1 \to \mathbb{CP}^1$ of degree one must be a Möbius transformation. It implies that, for any point $p \in M$, the preimage of (J_0, p) is a single $\mathrm{PSL}_2(\mathbb{C})$-orbit in $\mathcal{M}_A \times \mathbb{CP}^1$. Once we outrageously pretend that everything is finite dimensional, this and regularity of (J_0, p) imply (by degree theory, as sketched above) that $\overline{\pi \times \mathrm{ev}}$ is onto, as required.

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[1] Beware! In practice the domain and target of this map are infinite dimensional. In fact $\mathcal{J}(M,\omega)$ is a Fréchet manifold. I can quote a theorem in the literature that says that $\mathcal{M}_A$ is also a Fréchet manifold but at the moment I am not convinced of the proof that is provided there. Nevertheless, this *is* how we want to think of this map.

[2] Beware! Sard's theorem works for finite dimensional manifolds. There is also a version for Fredholm maps between Banach manifolds. There is a more recent version for Fréchet manifolds.

[3] For degree theory, see Chapter 4 of the book "Differential Topology" by Victor Guillemin and Alan Pollack; for infinite dimensional versions, see the book "Topics in nonlinear functional analysis" by Louis Nirenberg.

[4] Beware! In practice the proofs in the literature do something a bit different; see the references below.

Details of the actual argument that does work in this infinite dimensional setup can be found in the following books.

- "J-Holomorphic Curves and Symplectic Topology", by Dusa McDuff and Dietmar Salamon.
- "Holomorphic Curves in Symplectic Geometry". Editors: Michèle Audin and Jacques Lafontaine.
- "Symplectic Geometry; an introduction based on the seminar in Bern, 1992", by B. Aebischer, M. Borer, M. Kälin, Ch. Leuenberger, and H. M. Reimann.

Bibliography

[1] Bröcker and Jänich, "Introduction to Differential Topology".

[2] Ana Cannas da Silva, "Lectures on Symplectic Geometry", Springer-Verlag, Lecture Notes in Mathematics **1764**, revised 2006.

[3] Viktor Ginzburg, Victor Guillemin, and Yael Karshon, "Moment maps, cobordisms, and Hamiltonian group actions", American Mathematical Society, Mathematical surveys and monographs **98**, 2002.

[4] Victor Guillemin and Alan Pollack, "Differential Topology".

[5] Allen Hatcher, "Algebraic Topology".

[6] John Lee, *Introduction to Smooth Manifolds*, 2nd edition.

[7] Jürgen Moser, *On the volume elements on a manifofld*, Trans. Amer. Math. Soc. **120** (1965), 286–294.

[8] Michael Spivak, "A Comprehensive Introduction to Differential Geometry", Volume I.