Heegaard-Floer homology; a brief introduction

Jesse Frohlich

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1 Introduction

Heegaard-Floer homology, like most homology theories, takes in a geometric or topological structure and gives a (graded) abelian group. In this case, the geometric object is a closed, oriented 3-manifold Y.

A generalization can be applied to a knot K in a 3-manifold; the machinery of Heegaard-Floer homology is able to determine the genus of the knot (being the minimal genus of a surface with boundary K (after gluing a disk over the boundary)), which in particular can distinguish the unknot from all other knots.

2 Construction

To build the homology groups, we begin by expressing the 3-manifold in simpler terms:

Definition. A genus g handlebody is a 3-manifold M, whose boundary ∂M is homeomorphic to the genus g surface Σ_g (which is unique up to homeomorphism).

Our goal is to reduce our study to gluings of handlebodies. Given a 3-manifold Y, we want to decompose it as a union of handlebodies.

Definition. A Heegaard splitting of Y is a pair of handlebodies $U_0, U_1 \subseteq Y$ which are embedded submanifolds with boundary which have as a common boundary a genus-g surface $\Sigma_g \subseteq Y$.

Given a model genus g handlebody U, we can describe Y as a pair of embeddings $\varphi_i\colon U \hookrightarrow Y$. Up to homeomorphism, these embeddings are determined by how φ_i restricts to $\Sigma := \partial U$. Further $\varphi_i|_\Sigma$ is determined by a collection of curves in $\varphi_i(\Sigma)$ which are contractible in $\varphi_i(U)$. From this, we get:

Definition. A set of attaching circles for φ_i is a collection of g closed disjoint curves $\alpha_1, \dots, \alpha_g$ in Σ which satisfy:

- The space $\Sigma \setminus \bigcup_i \alpha_i$ is connected.
- The α_i are boundaries of disjoint disks in U.

Definition. A Heegaard diagram for Y is a genus g surface Σ together with two sets $\{\alpha_i\}$, $\left\{\beta_j\right\}$ of attaching circles for Σ so that the induced 3-manifold is homeomorphic to Y.

Up to a limited collection of transformations of these diagrams (namely isotopy, handle slides, and stabilization), the diagram uniquely determines a 3-manifold:

Theorem 1. *All 3-manifolds admit a Heegaard diagram. Furthermore, given two Heegaard diagrams for the same 3-manifold, there exist stabilizations of each which are equivalent under the aforementioned transformations.*

We now have a combinatorial description of our 3-manifold via a Heegaard diagram $(\Sigma_g, \alpha_i^0, \alpha_i^1)$. Next, we construct the symmetric space on Σ_g, $\mathcal{S}^g\left(\Sigma_g\right) = \Sigma_g^{\times g}/S_g$, which contain two (totally real) tori $\mathbb{T}_i := \alpha_1^i \times \cdots \times \alpha_g^i$.

Definition. Given two intersection points $\mathbf{x}$, $\mathbf{y} \in \mathbb{T}_0 \cap \mathbb{T}_1$ a Whitney disc is a map $\phi\colon \mathbb{D}^2 \to \Sigma_g$ such that $\phi(-\mathbf{i}) = \mathbf{x}$, $\phi(\mathbf{i}) = \mathbf{y}$, and $\phi(e_0) \subseteq T_0$ and $\phi(e_1) \subseteq T_1$, where e_0 denotes the portion of $\partial\mathbb{D}^2$ with nonpositive real part, and e_1 is that with nonnegative real part. We write $\mathrm{WD}(\mathbf{x}, \mathbf{y})$ to denote the space of all Whitney discs, considered up to homotopy.

We can a groupoid structure on $\mathrm{WD}(\mathbb{T}_0 \cap \mathbb{T}_1)$, given by gluing two discs together to form composition:

$$*\colon \mathrm{WD}(\mathbf{x}, \mathbf{y}) \times \mathrm{WD}(\mathbf{y}, \mathbf{z}) \to \mathrm{WD}(\mathbf{x}, \mathbf{z}) \tag{1}$$

Next, given a point $w \in \Sigma \setminus \bigcup_{i,j} \alpha_j^i$ away from the attaching circles, we build a locally constant map $n_w\colon \mathrm{WD}(\mathbf{x}, \mathbf{y}) \to \mathbb{Z}$ which counts the number of intersections with w, namely the *algebraic intersection number.* We then use the coefficients n_w to construct:

Definition. Let $\{D_i\}$ denote the set of closures of connected components of $\Sigma \setminus \bigcup_{i,j} \alpha_j^i$ (viewed as 2-chains in Σ_g). The domain associated to $\varphi \in \mathrm{WD}(\mathbf{x}, \mathbf{y})$ is given by:

$$\begin{aligned} \mathcal{D} \colon \mathrm{WD} &\to C_2(\Sigma_g) \\ \phi &\mapsto \sum_i n_{w_i}(\phi) D_i \end{aligned} \tag{2}$$

where $w_i \in D_i$. This defines a groupoid homomorphism.

Next, given $z \in \Sigma \setminus \alpha_j^i$, there is a map $s_z \colon \mathbb{T}_0 \cap \mathbb{T}_1 \to \mathrm{Spin}^{\mathrm{c}}(Y)$, whose definition relies on a Morse-theoretic interpretation of the attaching circles. The important fact is that there is a free and transitive group action $\mathrm{H}^2(Y, \mathbb{Z}) \circlearrowright \mathrm{Spin}^{\mathrm{c}}(Y)$.

Definition. Given $\phi \in \mathrm{WD}(\mathbf{x}, \mathbf{y})$, write the moduli space of holomorphic representatives of ϕ as $\mathcal{M}(\phi)$, then mod out by holomorphic reparametrizations to get $\widehat{\mathcal{M}}(\phi) = \mathcal{M}(\phi)/\mathbb{R}$. After an appropriate perturbation, $\widehat{\mathcal{M}}(\phi)$ can be endowed with a natural structure of an oriented 0-dimensional manifold. We denote by $c(\phi)$ the signed count of points in $\widehat{\mathcal{M}}(\phi)$.

We are now at the stage of defining the chain complex in Heegaard Floer homology:

Definition. Let Y be a 3-manifold. Given a Heegaard diagram with basepoint z and $\mathrm{Spin}^{\mathrm{c}}$-structure t on Y, namely $(\Sigma, \alpha_j^i, z, t)$, define the chain complex as

$$\widehat{\mathrm{CF}}(\alpha, t, z) \coloneqq s_z^{\mathrm{pre}}(t) = \left\{ \mathbf{x} \in \mathbb{T}_0 \cap \mathbb{T}_1 : s_z(\mathbf{x}) = t \right\} \tag{3}$$

With a relative grading defined in terms of the Maslov index μ (related to the "size" of $\widehat{\mathcal{M}}(\phi)$) and n_z. Define the differential as:

$$\partial \mathbf{x} = \sum_{\substack{\mathbf{y} \in \widehat{\mathrm{CF}}(\alpha, t, z) \\ \phi \in \mathrm{WD}(\mathbf{x}, \mathbf{y}) \cap n_z^{\mathrm{pre}}(0)}} c(\phi) \cdot \mathbf{y} \tag{4}$$

We then define the homology groups $\widehat{\mathrm{HF}}(Y, t)$ as build from the $\widehat{\mathrm{CF}}(\alpha, t, z)$ chain complex.

Remark 2. When $\mathrm{H}_1(Y, \mathbb{Z}) = 0$, then the choice of $\mathrm{Spin}^{\mathrm{c}}$-structure on Y is unique.

Remark 3. One can also define a slightly larger chain complex, $\mathrm{CF}^\infty(Y, t) \coloneqq \widehat{\mathrm{CF}}(Y, t) \times \mathbb{Z}$, which gives rise to another homology theory.

References

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