# AN INTRODUCTION TO GEOMETRIC QUANTIZATION

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## 1. Background

In classical Hamiltonian mechanics, the phase space may be interpreted as a symplectic manifold, e.g. $T^*\mathbb{R}^n$. The classical observables are smooth functions over the phase space. In quantum mechanics, the phase space is replaced by a Hilbert space, and quantum observables are self-adjoint operators on the Hilbert space.

A quantization program, roughly speaking, aims at designing a scheme that creates a quantum system from a given classical system. For example, the Weyl quantization (see e.g. [1]), designs a scheme that quantizes the classical phase space $T^*\mathbb{R}$ to a quantum Hilbert space $L^2(\mathbb{R})$ with the classical position function $x$ quantized to $f \mapsto xf$ and the classical momentum function $p$ quantized to $f \mapsto -\frac{i}{\hbar}\frac{d}{dx}f$.

This report gives a brief introduction to a quantization program named geometric quantization.

**Notice**: Throughout the report, our conventions for Hamiltonian vector fields and Poisson brackets and curvature of connections are

$$\imath(X_f)\omega = df, \quad \{f, g\} = \mathcal{L}_{X_f} g = -\omega(X_f, X_g), \quad X_{\{f,g\}} = [X_f, X_g],$$

$$F^\nabla(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

## 2. Prequantization

We start with the prequantization of a symplectic manifold $(M, \omega)$. More precisely, we shall construct a Hilbert space $\mathcal{H}_0$ associated with $M$ and a map assigning to each smooth real function on $M$ to a skew-symmetric operator on $\mathcal{H}_0$. The story begins with the following lemma.

**Lemma 2.1.** Let $M$ be a smooth manifold and $\omega$ a closed 2-form on $M$. If $\omega$ is integral, i.e. $[\omega] \in H^2_{dR}(M)_{\mathbb{Z}}$ then there exists a Hermitian line bundle $L \to M$ with a Hermitian connection $\nabla$ whose curvature satisfies $\frac{\sqrt{-1}}{2\pi}F^\nabla = \omega$. Here $H^2_{dR}(M)_{\mathbb{Z}}$ is the image of $H^2(M, \mathbb{Z})$ in $H^2_{dR}(M)$.

The idea of the proof is as follows ([3]).

Let $\{U_i\}$ be a good cover of $M$. On each $U_i$, $\omega = d\alpha_i$ for some $\alpha_i \in \Omega^1(U_i)$. On each $U_{ij} = U_i \cap U_j \neq \emptyset$, $\alpha_i - \alpha_j = df_{ij}$ for some $f_{ij} \in C^\infty(U_{ij})$. Note that $f_{ij} + f_{jk} - f_{ik}$ is constant since its differential vanishes. Now $[\omega] \in H^2(M, \mathbb{Z})$ implies that we can choose $\{f_{ij}\}$ such that $f_{ij} + f_{jk} - f_{ik} \in \mathbb{Z}$. Then we can define $c_{ij} := \exp(2\pi\sqrt{-1}f_{ij})$. Note that $c_{ij}c_{jk} = c_{ik}$, i.e. a cocycle. Thus $\{c_{ij}\}$ determines a complex line bundle $L$ over $M$.

Now consider a family of one forms $\{\theta_i := -2\pi\sqrt{-1}\alpha_i\}$. It indeed defines a connection $\theta$ on $L$. In fact,

$$\theta_i + c_{ji} dc_{ji}^{-1} = -2\pi\sqrt{-1}\alpha_i + 2\pi\sqrt{-1} df_{ij} = -2\pi\sqrt{-1}\alpha_j = \theta_j$$

i.e. the compatibility holds.

Finally, gluing together the standard Hermitian metric on $U_i \times \mathbb{C}$ by $\{c_{ij}\}$, we obtain a Hermitian metric on $L$ such that $\theta$ is a Hermitian connection basically since $\theta$ is imaginary-valued.

In summary, we obtained a Hermitian line bundle $L$ with a Hermitian connection such that $\frac{\sqrt{-1}}{2\pi} F^\nabla = \omega$.

A symplectic manifold is said to be prequantizable if the symplectic form is integral and a Hermitian line bundle (with a Hermitian connection) constructed above is called a prequantum line bundle. Now let $(M, \omega)$ be a prequantizable symplectic manifold and $(L, h, \nabla)$ a prequantum line bundle. We introduce the following inner product over the space $\Gamma_c(L)$ of smooth sections of $L$ with compact supports:

$$\langle s_1, s_2 \rangle := \int_M h(s_1, s_2) \frac{\omega^n}{n!}.$$

**Definition 2.2** (Prequantum Hilbert space)**.** The prequantum Hilbert space $\mathcal{H}_0$ is the completion of $\Gamma_c(L)$ with respect to the above inner product.

**Example 2.3.** Let $M = T^*\mathbb{R}^n$ with coordinates $(q_1, \cdots, q_n, p_1, \cdots, p_n)$ equipped with the standard symplectic form $\omega_{can} = \sum_j dq_j \wedge dp_j$. Then $L := M \times \mathbb{C}$ is a prequantum line bundle with the standard Hermitian metric induced from $\mathbb{C}$ and $\nabla f := df + \theta f$ is a Hermitian connection on $L$ such that $\frac{\sqrt{-1}}{2\pi} F^\nabla = \omega_{can}$, where $\theta = 2\pi\sqrt{-1}\sum_j (p_j dq_j)$ and $f \in C^\infty(M, \mathbb{C})$. In this case, the prequantum Hilbert space of $T^*\mathbb{R}^n$ is $\mathcal{H}_0 = L^2(\mathbb{R}^{2n})$.

Next we want to prequantize the classical observables, i.e. smooth real-valued functions over $M$.

**Definition 2.4** (Prequantization map)**.** The prequantization map $Q : C^\infty(M, \mathbb{R}) \to \mathrm{End}(\Gamma_c(L))$ is defined by $Q_f = \nabla_{X_f} - 2\pi\sqrt{-1} f$.

**Lemma 2.5** ([4])**.** The prequantization map $Q_f$ is skew-symmetric with respect to $\langle ., . \rangle$ and $[Q_f, Q_g] = Q_{\{f,g\}}$, i.e. $Q$ is a Lie algebra homomorphism from $C^\infty(M, \mathbb{R})$ to $\mathrm{End}(\Gamma_c(L))$.

*Proof.* (1): For any two sections $s_1, s_2$ of $L$,

$$\begin{aligned}\langle Q_f(s_1), s_2 \rangle + \langle s_1, Q_f(s_2) \rangle &= \langle \nabla_{X_f} s_1 - 2\pi\sqrt{-1} f s_1, s_2 \rangle + \langle s_1, \nabla_{X_f} s_2 - 2\pi\sqrt{-1} f s_2 \rangle \\ &= \int_M (h(\nabla_{X_f} s_1, s_2) + h(s_1, \nabla_{X_f} s_2)) \frac{\omega^n}{n!} = 0.\end{aligned}$$

The last equality holds since $h(\nabla_{X_f} s_1, s_2) + h(s_1, \nabla_{X_f} s_2) = X_f(h(s_1, s_2))$ and

$$\int_M X_f(h(s_1, s_2)) \frac{\omega^n}{n!} = \int_M \mathcal{L}_{X_f} \left( h(s_1, s_2) \frac{\omega^n}{n!} \right) = \int_M d\imath(X_f) \left( h(s_1, s_2) \frac{\omega^n}{n!} \right) = 0$$

by Stokes' theorem.

(2): Note that

$$\nabla_{X_f}\nabla_{X_g} - \nabla_{X_g}\nabla_{X_f} - \nabla_{[X_f,X_g]} = 2\pi\sqrt{-1}\{f,g\}$$

by expanding the formula $F^{\nabla}(X_f,X_g) = -2\pi\sqrt{-1}\omega(X_f,X_g) = 2\pi\sqrt{-1}\{f,g\}$. Now $[Q_f,Q_g] = Q_{\{f,g\}}$ follows from straightfoward calculations:

$$\begin{aligned}[Q_f,Q_g] =& \nabla_{X_f}\nabla_{X_g} - \nabla_{X_g}\nabla_{X_f} - 2\pi\sqrt{-1}X_f(g) + 2\pi\sqrt{-1}X_g(f)\\ =& \nabla_{[X_f,X_g]} + 2\pi\sqrt{-1}\{f,g\} - 2\pi\sqrt{-1}\{f,g\} - 2\pi\sqrt{-1}\{f,g\}\\ =& \nabla_{X_{\{f,g\}}} - 2\pi\sqrt{-1}\{f,g\}.\end{aligned}$$

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## 3. Polarizations

It turns out that a prequantum Hilbert space is too large. For example, the prequantum Hilbert space of $T^*\mathbb{R}^n$ is $L^2(\mathbb{R}^{2n})$. But physicists tell us that the quantum Hilbert space should be $L^2(\mathbb{R}^n)$, the space of wave functions.

In this section, we will introduce one more step which enables us to get a right "size" quantum space from the prequantum Hilbert space. The tool we need is polarizations.

**Definition 3.1** (Polarization)**.** Let $(M,\omega)$ be a symplectic manifold. A (complex) polarization $P$ of $M$ is an involutive Lagrangian subbundle of $(TM^{\mathbb{C}},\omega^{\mathbb{C}})$ with the rank of $P\cap\bar{P}$ being constant.

**Example 3.2.** (1) (Vertical polarization [1]) For any smooth manifold $N$, the cotangent bundle $M := T^*N$ is a symplectic manifold with the canonical 2-form $\omega_{\text{can}}$ and a projection $\pi : M \to N$. Then $P_z := T^*_{\pi(z)}N^{\mathbb{C}} \subset T_zM^{\mathbb{C}}$ forms a polarization of $M$ with $P = \bar{P}$.

(2) (Purely complex polarization [1]) Let $(M,J)$ be a complex manifold. Then the $(-\sqrt{-1})$-eigenbundle $P$ of $J$ is a polarization of $M$ with $P\cap\bar{P} = \{0\}$. In particular, if $(M,\omega,J)$ is Kähler, we call $P$ a Kähler polarization.

(3) [4] Let $M := \mathbb{R}^2 - \{0\}$. Then $P := \text{Span}_{\mathbb{C}}(\frac{\partial}{\partial\theta})$ is a polarization of $M$, where $(r,\theta)$ is the polar coordinates of $M$.

Now we proceed our discussion of quantizing a symplectic manifold $(M,\omega)$. Suppose $M$ is prequantizable, $L$ is a prequantum line bundle over $M$ and $P$ is a polarization of $M$.

**Definition 3.3.** Let $\Gamma_P(L)$ denote the space of polarized sections of $L$, i.e which are convariantly constant along the directions of $P$:

$$\Gamma_P(L) = \{s \in \Gamma(L) : \nabla_X s = 0 \,\text{for all}\, X \in \Gamma(P)\}.$$

**Example 3.4.** Let us continue Example 2.3. Let $P$ be the vertical polarization of $T^*\mathbb{R}^n$. It turns out that $\Gamma_P(L) = \{f \in C^\infty(\mathbb{R}^{2n}) : f(q,p) = f(q,p'), \forall p,p' \in \mathbb{R}^n\}$.

**Remark 3.5.** If the polarization we choose is not so good, then the space $\Gamma_P(L)$ may be zero. E.g. this is the case in part (3) of Example 3.2.

Even if the space $\Gamma_P(L)$ is nontrivial, in general, it is not so easy to make it into a Hilbert space. One way to solve this issue is by doing the half-form correction (ref. [1], [4]).

**Example 3.6** ([1])**.** Let $(M, \omega, J)$ be a Kähler manifold and $P$ the Kähler polarization of $M$. Assume $\omega$ is integral and $(L, h, \nabla)$ is a prequantum line bundle. With the above datum, $L$ has a holomorphic structure such that the restriction of $\nabla$ to $P$ is $\bar{\partial}_L$ and $\Gamma_P(L)$ is the space of holomorphic sections of $L$ (see e.g. Theorem 23.31 in [1]).

By Hodge theory, $\Gamma_P(L)$ is a finite-dimensional vector space if $M$ is compact [2]. Moreover, by Kodaira's embedding theorem [2], for sufficiently large $k$, the holomorphic line bundle $L^{\otimes k}$ has a global nonzero holomorphic section. Thus if $M$ is a symplectic manifold equipped with the symplectic form $k\omega$, then $L^{\otimes k}$ is a prequantum line bundle and $\Gamma_P(L^{\otimes k})$ is nonzero. Due to this reason, geometric quantization works well for compact Kähler manifolds.

**Example 3.7** (Quantization of $\mathbb{C}P^n$)**.** Let $M = \mathbb{C}P^n$, $\omega = \omega_{FS}$, the normalized Fubini-Study form and $P$ be the Kähler polarization.

The hyperplane bundle $\mathcal{O}(1)$ has global holomorphic sections $z_0, \cdots, z_n$ (coordinates of $\mathbb{C}^{n+1}$) and they induce a Hermitian metric $h$ on $\mathcal{O}(1)$ by $h(s,s) = \frac{|\psi(s)|^2}{\sum_j |\psi(z_j)|^2}$, where $\psi$ is any local trivialization of $\mathcal{O}(1)$. (see e.g. [2]).

Then the hyperplane bundle $L = \mathcal{O}(1)$ with the Chern connection is a prequantum line bundle. The space $\Gamma_P(L)$ is the global holomorphic sections of $\mathcal{O}(1)$, i.e. complex polynomials in $n+1$ variables of degree one.

We conclude our report by quantizing classical observables. We say a function $f \in C^\infty(M, \mathbb{R})$ is polarization-preserving or quantizable if $[X_f, X] \in \Gamma(P)$ for any $X \in \Gamma(P)$. Let $C^\infty_P(M)$ be the space of quantizable functions over $M$.

**Theorem 3.8** ([4])**.** *The algebra $C^\infty_P(M)$ is a Lie subalgebra of $C^\infty(M)$ with respect to the Poisson bracket. In addition, $Q(C^\infty_P(M)) \subset \mathrm{End}(\Gamma_P(L))$ and the map $Q|_{C^\infty_P(M)} : C^\infty_P(M) \to \mathrm{End}(\Gamma_P(L))$ is a Lie algebra homomorphism.*

*Proof.* (1): For any $f, g \in C^\infty_P(M)$, Using the fact that $X_{\{f,g\}} = [X_f, X_g]$ and Jacobi's identity, we see that

$$[X_{\{f,g\}}, X] = [[X_f, X], X_g] + [X_f, [X_g, X]] \in \Gamma(P).$$

(2):Let $f \in C^\infty_P(M)$ and $s \in \Gamma_P(L)$. Then

$$\nabla_X(Q_f s) = \nabla_X(\nabla_{X_f} s - 2\pi\sqrt{-1} f s) = \nabla_X \nabla_{X_f} s - 2\pi\sqrt{-1}(Xf)s - 2\pi\sqrt{-1} f \nabla_X s.$$

The last term on the right hand side vanishes since $s \in \Gamma_P(L)$. The first two terms on the right hand side cancel with each other since $\nabla_X \nabla_{X_f} s = -2\pi\sqrt{-1}\omega(X, X_f)s = 2\pi\sqrt{-1}(Xf)s$ by investigating the formulas $F^\nabla(X, X_f)s = -2\pi\sqrt{-1}\omega(X, X_f)s$ and $\imath(X_f)\omega = df$.

(3): By Lemma 2.5, we conclude that $Q|_{C_P^\infty(M)}$ is a Lie algebra homomorphism. ■

## Reference

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