GROMOV'S COMPACTNESS THEOREM

REILA ZHENG

1. Notation and Definitions

In this paper, "manifold" always refers to smooth, finite-dimensional manifold. By "surface" we mean a dimension 2 manifold. By "closed" we mean a compact manifold with empty boundary.

We fix the following notation for the rest of this paper. Denote (M, J, g) an almost-complex manifold of $\mathbb{R}$-dimension $2n$, J an almost-complex structure, and g a Hermitian Riemannian metric on M. Let (S, j) be a Riemann surface with j a complex structure.

We say $f : (M, J) \to (S, j)$ is *J-holomorphic* if $df \circ j = J \circ df$ (we will say f is holomorphic if the context is clear). Note that we will generally drop the type signature if the context is clear.

We say $N \subseteq M$ is a *totally real submanifold* if N is a submanifold of $\mathbb{R}$-dimension n and $J(T_pN) \cap T_pN = \{0\}$ for every $p \in N$. We will denote N a totally real submanifold of M.

1.1. Riemannian Geometry.

We first recall some definitions and facts from Riemannian geometry.

A *Riemann metric* g on M is a Euclidean scalar product g_p on T_pM at every point $p \in M$ that depends smoothly on p, i.e., for any X, Y vector fields on M, $g(X, Y) : p \mapsto g_p(X(p), Y(p))$ is smoooth. A *Riemannian manifold* (M, g) is a manifold equipped with a Riemannian metric.

We have a norm given by g, such that for any $p \in M$, $||v||_g := \sqrt{g(v, v)}$ for $v \in T_pM$.

For the rest of this paper let γ denote a smooth path $\gamma : [a, b] \to M$. Then its *length* is given by $l(\gamma) := \int_a^b ||\dot{\gamma}(t)||_g dt$. We can equip (M, g) with a distance $d_g(p, q) := inf\{l(\gamma)|\gamma : [0, 1] \to M$ is a smooth path from p to $q\}$, where p, q are in the same connected component of M (we can define the distance to be ∞ if they are not path-connected).

Note that any manifold M also embeds into $\mathbb{R}^N$ for some large N so we also have the Euclidean distance on the inclusion of M. For the rest of this paper I will denote $B_r(x)$ to be the open ball centered at x of radius r in the Riemannian distance, and and $D_r(x)$ the ball in the Euclidean distance (for some fixed embedding of $M \subseteq \mathbb{R}^N$).

We say γ is a *geodesic* if it has constant speed and is locally distance-minimizing for the Riemannian distance. Then by the local existence and uniqueness of solutions to this ODE, for any $\gamma(0) = p \in M$ and $\dot{\gamma}(0) = v \in T_pM$ there is a unique maximal domain $I_v \ni 0$ such that $\gamma : I_v \to M$ is a geodesic. Denote $D := \{(p, v) \in TM | 1 \in I_v\}$ a neighbourhood of the 0 section in TM and $exp : D \to M$ the *exponential map* such that $exp(p, v) := \gamma(1)$ where γ is the geodesic satisfying $\gamma(0) = p$ and $\dot{\gamma}(0) = v$.

Note that at every point $(p, v) \in D$, exp is a local diffeomorphism. We can define the *injectivity radius at a point* $p \in M$ by $inj(p) := \sup\{r | exp$ is a local diffeomorphism on $\{p\} \times B_r(0)\}$. Note that since exp is a local diffeomorphism $inj(p) > 0$. We define the *injectivity radius of the manifold* to be $inj(M) := \inf\{inj(p) | p \in M\}$. Note that if M is compact then $inj(M) > 0$.

For a Riemannian manifold the *sectional curvature at a point* is a map $K_p : \{V \subseteq T_pM | \dim(V) = 2\} \to \mathbb{R}$. For $p \in M$ and a fixed $V \subseteq T_pM$ a dimension 2 subspace, we can let $M_V \subseteq M$ to be the subspace of M traversed by geodesics with velocity in V at the point p. Then M_V is a dimension 2 local submanifold of M. Note that $V = T_pM_V$, and the curvature K_p on V is the Gaussian curvature. We denote the *sectional curvature* $K : \{(p, V) \subseteq TM | \dim(V) = 2\} \to \mathbb{R}$ by $K : (p, V) \mapsto K_p(V)$. Note that if M is a compact manifold then K is uniformly bounded.

We say that M has *bounded geometry* if M has positive injectivity radius, uniformly bounded sectional curvature, and uniformly bounded Levi-Civita connection. (Where uniformly bounded means uniform over $p \in M$ as an operator.)

For the rest of this paper we will assume that M has bounded geometry.

1.2. **Riemann Surface with Nodes.** We now define a Riemann surface with nodes. The following notation will be used throughout the rest of this subsection.

Let (S, j) be a closed Riemann surface, and let $\{\gamma_i\}_{i \in I}$ be a set of finitely many simple smooth curves $\gamma_i : S^1 \to S$, with pairwise disjoint image diffeomorphic to S^1. Denote $\hat{S}$ the compactification of $S \setminus \cup_{i \in I} \gamma_i(S^1)$ by attaching two points, x_i and x_i' to compactify each of the sides after removing γ_i. Then the complex structure j on S descends to a structure (also denoted by j) on $\hat{S}$. Let $\bar{S}$ be $\hat{S}/x_i \sim x_i'$, the space obtained from S after identifying the image of each γ_i to a point s_i, and denote $\alpha : \hat{S} \to \bar{S}$ the quotient map. We will endow $\bar{S}$ with a complex structure $\bar{j}$ inherited from $\hat{S}$. We say $(\bar{S}, \bar{j})$ is a *Riemann surface with nodes*.

In the above construction we say the points $s_i \in \bar{S}$ such that $s_i = \alpha(\gamma_i(S^1))$ for $i \in I$ are *singular points* or *nodes*. Denote $si(\bar{S})$ the set of all the singular points in $\bar{S}$.

We now define a node map. Let $(S, j), (S', j')$ be two Riemann surfaces with nodes, and suppose $\phi : S \to S'$ is continuous, surjective, and satisfies the following:

(1) For every node $x \in si(S')$, $\phi^{-1}(x)$ is either a node, a simple closed curve in $int(S)$, or if S' has boundary, a simple closed arc with endpoints in ∂S.

(2) ϕ is a diffeomorphism outside of the set of singular points, that is $\phi|_{S \setminus \phi^{-1}(si(S'))} : S \setminus \phi^{-1}(si(S')) \to S' \setminus si(S')$ is a diffeomorphism in the usual sense. (It is not clear in Hummel or Ye's statement but Gromov's original paper seems to support this, in it he describes the pinching of geodesics to points via homeomorphisms.)

Then we say ϕ is a *node map*. The idea is that ϕ may collapse simple curves to points, but it does not otherwise disturb the topological structure.

We say that a map $f : (\bar{S}, \bar{j}) \to (M, J)$ is a *cusp curve* if f is continuous and $f \circ \alpha : (\hat{S}, j) \to (M, J)$ is J-holomorphic. We say the *area* $A(f)$ is $A(f \circ \alpha)$.

We say that f is cusp curve with *boundary* in a totally real submanifold $N \subseteq M$ if the image of ∂S is contained in N.

We say the *genus* of a Riemann surface with nodes $\bar{S}$ is given by $genus(\bar{S}) = genus(\hat{S})$. We say the *connectivity* of $\bar{S}$ is $conn(\bar{S}) = conn(\hat{S})$, the number of boundary components. We say the *homology class* of a cusp curve f is given by the homology class of $f \circ \alpha$.

Note that after applying a node map the number of nodes is increasing, the genus is decreasing, and the connectivity is increasing.

We now put a topology on the space of compact cusp curves with image in (M, J). Let $f : S \to M$ be a compact cusp curve (i.e., S is compact). Given $\varepsilon > 0$, a metric on S, a neighbourhood $U \subseteq S$ of $si(S) \subseteq U$, we can define a neighbourhood $F \ni f$ as follows. For any compact cusp curve $\tilde{f} : (\tilde{S}, \tilde{j}) \to (M, J)$, $\tilde{f} \in F$ if there exists $\phi : S \to \tilde{S}$ a node map such that the following 3 conditions hold:

(1) $||f - \tilde{f} \circ \phi^{-1}||_{C^k} < \varepsilon$ on $S \setminus U$.
(2) $||j - \phi^* \tilde{j}||_{C^k} < \varepsilon$ on $S \setminus U$.
(3) $|A(f) - A(\tilde{f})| < \varepsilon$.

In the above definition, $||.||_{C^k}$ is the C^k-norm, it and $A(.)$ are given from the metric on M and S.

We call the topology generated by these neighbourhoods the *C^k-topology* on the space of compact cusp curves.

We denote $\mathcal{C}$ to be the set of compact cusp curves with image in (M, J), and $\mathcal{F}$ the set of closed (i.e., the domain is a closed manifold) cusp curves, $\mathcal{F} \subseteq \mathcal{C}$.

By definition, convergence in C^∞ means convergence in C^k for all $k \geq 0$. Note that in certain cases, convergence in C^k is equivalent for all k so there is no ambiguity in writing $\bar{\mathcal{F}}$ for its closure. This is true for the set of closed cusp curves and the set of cusp curves with boundary in N, which is the assumption of the theorem.

Let $m \in \mathbb{N}_0$ and $a > 0$. Denote $\mathcal{F}_{m,a}$ the set of all $f \in \mathcal{F}$ such that $genus(f) = m$ and $A(f) \leq a$. Moreover let $m' \in \mathbb{N}_0$ and $N \subseteq M$ a totally real submanifold. Denote $\mathcal{F}_{m,m',a}(N)$ the set of all $f \in \mathcal{C}$ such that $f(\partial S) \subseteq N$ and $genus(f) = m$, $conn(f) = m'$, and $A(f) \leq a$.

1.3. **PDE Theory.** For $f \in L^p(U)$ where U is a domain, $\alpha \in \mathbb{N}^n$ a multi-index and denote $|\alpha| := \sum_{i=1}^n |\alpha_i|$, the *weak derivative* $D^\alpha f \in L^1_{loc}(U)$ satisfies $\forall \psi \in C_c^\infty(U)$ we have that $\int_U \psi D^\alpha f dx = (-1)^{|\alpha|} \int_U f D^\alpha \psi dx$, i.e., integration by parts holds. Weak derivatives, when they exist are unique and agree with the usual derivative when that exists.

For $k \in \mathbb{N}$ the *Sobolev space* $H^k(U) := \{f \in L^2(U) | D^i f \in L^2(U) \forall |i| \leq k\}$, the norm is given by $||f|| := (\sum_{|i| \leq k} ||D^i f||_2^2)^{1/2}$. Denote $H^1(S, M)$ the Sobolev space of functions $f : S \to M$ where integration is taken over S and is dependent on g.

We say $f \in H^1(S, M)$ is a *weak holomorphic curve* if $J \circ df = df \circ j$.

For $1 \leq p < \infty, \lambda \geq 0$, the *Morrey space* $L^{p,\lambda}(U) := \{f \in L^p(U) | \exists B \forall x_0 \in U, r > 0, \int_{D_r(x_0) \cap U} |f|^p dx \leq B^p r^\lambda\}$. The norm is given by $||f|| := (\sup\{x_0 \in U, r > 0 | r^{-\lambda} \int_{D_r(x_0) \cap U} |f|^p dx\})^{1/p}$.

For $1 \leq p < \infty, \lambda \geq 0$ the *Campanato space* $\mathcal{L}^{p,\lambda}(U) := \{f \in L^p(U) | [f]_{p,\lambda} < \infty\}$ where the seminorm $[f]_{p,\lambda} := (\sup\{x_0 \in U, r > 0 | r^{-\lambda} \int_{D_r(x_0) \cap U} |f - \bar{f}_{r,x_0}|^p dx\})^{1/p}$, where $\bar{f}_{r,x_0} := \int_{D_r(x_0) \cap U} f / A(D_r(x_0) \cap U)$ the average of the value of f in the ball.

2. Statement of Theorem

The methods in Ye's paper are used to prove two results. First is the usual Gromov's Compactness Theorem for closed holomorphic curves. This says essentially that a sequence of closed holomorphic curves with the same topological genus and uniformly bounded in area converges to a cusp curve, in the sense of smooth convergence outside of the singular points and convergence in area.

The second theorem is Gromov's Compactness Theorem for holomorphic curves Boundary. This says that the same conclusion holds for if additionally we fix some totally real submanifold $N \subseteq M$, and if the sequence of compact holomorphic curves with the same genus and uniform bound in area also satisfies the additional criteria that they have the same connectivity and the image of the boundary of S is in N.

Ye also claimed both results are still true if J is allowed to range through a family of submanifolds parametrized by a compact set, and the second result holds if N is allowed to range through a compact set of submanifolds. We now state the theorems.

Theorem 1 (GCT for Closed Pseudo-Holomorphic Curves)**.** *For any (M, J, g) and $\mathcal{F}_{m,a}$ defined for such (M, J, g), $\bar{\mathcal{F}}_{m,a}$ is sequentially compact.*

We use notation as before. Note that if a sequence of cusp curves converge in the topology above then eventually the genus of the cusp curves will be eventually constant. Thus if (S_∞, j_∞) is the Riemann surface at the limit f_∞ of the sequence f_k on (S_k, j_k), we have that the definition of the topology C^∞ implies the existence of the node maps $\phi_k : S_k \to S_\infty$. Condition (1) implies the cusp curves $f_k \circ \phi_k^{-1}$ converge to f_∞ in C^∞ uniformly on compact sets outside of the singular points of S_∞. Condition (2) implies that the structures $\phi_k^* j_k$ converge to j_∞ in C^∞ uniformly on compact sets outside of the singular points of S_∞. Condition (3) implies there is no area loss, $A(f_k) \to A(f_\infty)$.

Theorem 2 (GCT for Pseudo-Holomorphic Curves with Boundary)**.** *For any (M, J, g) and N a totally real submanifold, and $\mathcal{F}_{m,m',a}(N)$ defined for such (M, J, g) and N, $\bar{\mathcal{F}}_{m,m',a}(N)$ is sequentially compact.*

3. Discussion of the Proof

In Gromov's original proof of the result for closed holomorphic curves the two main lemmas used were the Gromov-Schwarz Lemma, and an Isoperimetric Inequality. The former is a generalization of the classical Schwarz Lemma which says for a holomorphic function on the disk, $f : D \to D$ such that f fixes 0, then $|f'(0)| \leq 1$. Gromov extended this result to the case where (M, J, g) is compact, then there is some $\varepsilon(M) > 0, C(M) > 0$ such that whenever $f : \mathbb{H} \to M$ is holomorphic with $f(\mathbb{H})$ contained in some ε-ball in M must satisfy $||df|| \leq C(M)$, where the norm on df is the norm with respect to g on the tangent space, and $\mathbb{H}$ is the half-plane (equivalently the disk by a mobius map).

An Isoperimetric Inequality relates the length of curve on the boundary of a surface to the area of that surface. A classical example of this says whenever S has nonempty boundary and $\gamma : \partial S \to \mathbb{R}^m$ is a smooth curve, then γ extends to a smooth map f with domain S such that $4\pi A(f) \leq l(\gamma)^2$.

The same two results are used in Ye's proof, where the Gromov-Schwarz Lemma is obtained as part of some regularity results on weak holomorphic curves. This is the interior regularity argument, which upgrades the regularity of a weak holomorphic closed curve to a continuous map. By standard results from nonlinear elliptic PDEs continuous can be upgraded to Holder-continuous and C^1. The results used here relies on properties of Morrey and Campanato spaces. Iteratively repeating the argument allows Ye to obtain smoothness.

The boundary regularity argument treats the case of curves with boundary. Ye repeats uses essentially the same argument on the interior, then chooses a suitable metric g_0 on M such that $J(T_pN) \perp T_pN$, i.e. N is pseudo-Lagrangian. This is done by taking a basis $\{e_1(p), \dots, e_n(p), Je_1(p), \dots Je_n(p)\}$ for $p \in N$ and letting g be a metric for which this basis is orthonormal. Let $\tilde{N} \supseteq N$ be a closed submanifold and extend g to a tubular neighbourhood of $\tilde{N}$. Interpolating $g(J., J.) + g$ with g Ye obtains a g_0 that is is Hermitian on M, and this g_0 satisfies the required property. Then Ye obtains the required by choosing suitable local coordinates on $\tilde{N}$ and applying the same strategy for nonlinear elliptic PDEs as in the argument for interior regularity.

The Isoperimetric Inequality follows from choosing a suitable g_0 as above and applying a result by Hartman-Wintner. Then Ye used the Sacks-Uhlenbeck argument for removing isolated singularities and a ε-regularity theorem to obtain the result. This argument uses the fact that a holomorphic curve into (M, g_0) is harmonic and conformal, so the argument of Gruter's regularity theorem for minimal surfaces applies. We will not delve into the details of this argument.

The advantage of Ye's proof over Gromov's original argument is that Ye's approach also works for holomorphic curves with boundary. Gromov did not prove the result for curves with boundary in the general case, and his idea for extending the result in this case relies on the reflection principle which only works when the boundary has nice enough geometry. Ye's argument for the boundary case relies on the symplectic structure which gives rise to N being a Lagrangian submanifold with respect to a suitable metric g_0. This condition is needed since Ye's argument uses a theorem of Y. G. Oh which allows for the removal of isolated singularities at a Lagrangian boundary.

References

[1] Gromov, M. *Pseudo holomorphic curves in symplectic manifolds*. Invent Math (1985) 82: 307. https://doi.org/10.1007/BF01388806

[2] Ye, R. *Gromov's compactness theorem for pseudo holomorphic curves*. Trans Am Math Soc (1994) 342:2. https://doi.org/10.1090/S0002-9947-1994-1176088-1

[3] Hummel, C. *Gromov's compactness theorem for pseudo-holomorphic curves*. Springer Basel AG, 1997.

[4] Lee, J. *Riemannian manifolds: an introduction to curvature*. Springer, 1997

[5] Rafeiro H., Samko N., Samko S. (2013) *Morrey-Campanato Spaces: an Overview*. Operat Th Adv Appl (2013) vol 228. https://doi.org/10.1007/978-3-0348-0537-7_15