The Earthquake Goes Where the Earthquake Flows

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Abstract

We discuss the earthquake flow about a simple closed curve α in the context of the algebraic formalism introduced by Goldman to extend the Weil-Petersson symplectic form to the $\mathrm{PSL}_2\,\mathbb{R}$ character variety. We outline the background and ideas behind Goldman's argument that earthquake flows are Hamiltonian with respect to the length functions on Teichmüller space.

1 Teichmüller space

1.1 Basic definitions

Let S_g be a closed oriented surface of genus $g \geq 2$. The Teichmüller space is the deformation space of hyperbolic structures on S_g and a point of Teichmüller space corresponds to S_g equipped with a hyperbolic structure defined up to isotopy in the following precise sense.

Definition 1. *A* marked hyperbolic structure *on S_g is a (complete) oriented hyperbolic surface (i.e. equipped with a constant curvature -1 Riemannian metric), X, equipped with an (orientation preserving) homeomorphism $\phi : S_g \to X$. The* Teichmüller space *of S_g is the space of marked hyperbolic structures up to homotopy,*

$$\mathcal{T}(S_g) := \{(X, \phi)\}/\sim$$

where $(X, \phi) \sim (Y, \psi)$ iff there exists an isometry $f : X \to Y$ such that $f \circ \phi$ is homotopic to ψ.

The uniformization theorem tells us we may realize any hyperbolic surface X as a quotient $X = \mathbb{H}/\Gamma$ where $\Gamma \leq \mathrm{PSL}_2\,\mathbb{R}$ is a discrete subgroup of isometries acting freely and properly discontinuously. This yields the alternative description of Teichüller space. Covering space theory tells us that $\mathbb{H}/\Gamma$ and $\mathbb{H}/g\Gamma g^-1$ carry isomorphic hyperbolic structures for any $g \in \mathrm{PSL}_2\,\mathbb{R}$.

Definition 2. *The Teichmüller space of S_g is the space of discrete and faithful representations of the fundamental group of S_g into the isometry group of $\mathbb{H}$ defined up to conjugation*

$$\left\{\rho : \pi_1(S_g) \xrightarrow[\textit{faithful}]{\textit{discrete}} \mathrm{PSL}_2\,\mathbb{R}\right\} / \,\mathrm{PSL}_2\,\mathbb{R}$$

See [2] section 10.1 for more details on definition 1 and section 10.3 for details on definition 2 and how one obtains a topology on Teichmüller space. In fact, $\mathcal{T}(S_g)$ has the structure of a complex manifold.

Remark What we have described is better referred to as *Fricke-Klein* space. This name emphasizes the fact we are dealing with hyperbolic structures whereas it is understood that we are studying complex structures when we use the term Teichmüller space. Often times, we don't distinguish the two since the uniformization theorem produces a bijection between the two. People who study both structures simultaneously do make the distinction and the identification is not as nice as one might hope.

1.2 Fenchel-Nielsen Coordinates

A *pairs of paints* is a disk minus two disjoint disks. A *pants decomposition* of a surface S_g is a collection

$$\mathcal{P} = \{\alpha_1, \ldots, \alpha_{3g-3}\}$$

of disjoint simple closed curves such that $S_g \setminus \cup\alpha_j$ is a disjoint union of pairs of paints.

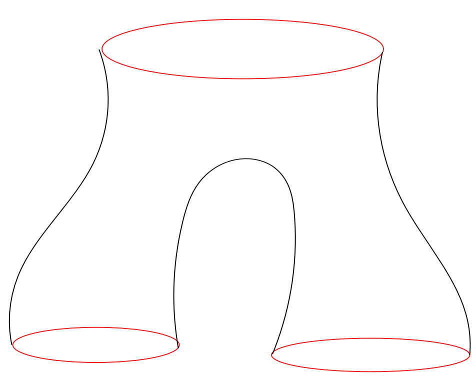

Figure 1: When shopping for pair of hyperbolic pants, the usual measurements don't tell you if they'll fit. Instead, the lengths of the three boundary curves tell you all you need to know about the geometry of your pants (and hence if they will fit you). It is generally good advice to purchase pants that fit.

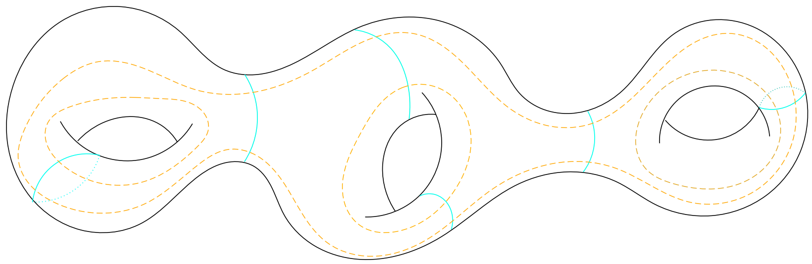

Figure 2: The collection of solid blue curves are a pants decomposition of the genus 3 surface pictured above. In dashed orange are seams for this pants decomposition.

Fix a pants decomposition $\mathcal{P}$ of S_g. For each curve $\alpha \in \mathcal{P}$, there is an associated length function

$$\ell_\alpha : \mathcal{T}(S_g) \to \mathbb{R}_+$$

which maps the hyperbolic structure X to the length of the geodesic representative of α in X. Each loop has a unique geodesic representative, so this function is well defined. The hyperbolic structure on a pair of pants is uniquely determined by the lengths of its boundary curves. The hyperbolic structure X is, however, not determined uniquely by these lengths. We can, in fact, recover a unique hyperbolic structure if we record the amount of "twisting" around each curve when regluing the pants. To define the twisting precisely, we require an additional set of disjoint simple closed curves on the surface called *seams* whose union intersect any pants determined by $\mathcal{P}$ in 3 arcs connecting the boundary components of the pants. The seams allow us to define the twist coordinate about a curve in the pants decomposition

$$\tau_\alpha : \mathcal{T}(S_g) \to \mathbb{R}$$

.

Theorem 1. *The map*

$$\begin{aligned} \mathcal{T}(S_g) &\to \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3} \\ X &\mapsto \big(\ell_{\alpha_1}(X), \ldots, \ell_{\alpha_{3g-3}}(X), \tau_{\alpha_1}(X), \ldots, \tau_{\alpha_{3g-3}}(X)\big) \end{aligned}$$

is a homeomorphism. The length and twist functions are called Fenchel-Nielsen coordinates. *They are in fact real analytic functions.*

See [2] section 10.5 for more on pants and section 10.6 for details on these coordinates (in particular for how the twist coordinates is defined).

2 The Weil-Petersson Metric

Teichmüller space carries a natural symplectic form called the Weil-Petersson form. We discuss serveral equivalent definitions.

André Weil introduced a Hermitian metric on Teichmüller space. By identifying the cotangent space of Teichmüller space at X with the space of quadratic differentials on X, $T_X\mathcal{T}(S_g) \cong QD(X)$, we obtain the Weil-Petersson cometric by integrating a pair of quadratic differentials against the hyperbolic area form, dA, on X,

$$\langle \phi, \psi \rangle = \int_X \phi \overline{\psi} dA$$

This in fact defines an incomplete Kähler and hence a symplectic form called the *Weil-Petersson symplectic form*, obtained by taking the imaginary part of the metric defined by this cometric. Teichmüller space is Weil-Petersson convex and has negative sectional curvature with respect to this metric. From this formulation of the metric, it can be seen that the Weil-Peterssen metric is mapping class group invariant and hence descends to the moduli space $\mathcal{M}_g = \mathcal{T}(S_g)/\operatorname{MCG}(S_g)$. [8]

In line with our second definition of $\mathcal{T}(S_g)$, Goldman introduced a symplectic form on the character variety $\operatorname{Hom}(\pi_1, \operatorname{PSL}_2\mathbb{R})/\operatorname{PSL}_2\mathbb{R}$ extending the Weil-Petersson symplectic form.

The tangent space space at a (representative of a) representation ρ can be identified with the 1^{st} cohomology group of S with coefficients in the Lie algebra $\mathfrak{sl}_2\mathbb{R}$ of $\operatorname{PSL}_2\mathbb{R}$ twisted by the adjoint representation of ρ, $\operatorname{Ad}\rho : \pi_1 \to \mathfrak{sl}_2\mathbb{R}$. This cohomology group, notated $H^1(S, \operatorname{Ad}\rho)$, can be described as follows. Let $\widetilde{S}$ be the universal cover of S_g and make $C_n(\widetilde{S}, \mathbb{Z})$ into a $\mathbb{Z}[\pi_1]$–module by composing a singular chain with the action π_1 on $\widetilde{S}$. The twisted cohomology groups are defined as the homology of the complex

$$C^*(S, \operatorname{Ad}\rho) = \operatorname{Hom}_{\mathbb{Z}[\pi_1]}(C_*(\widetilde{S}, \mathbb{Z}), \mathfrak{sl}_2\mathbb{R})$$

There is a cup product on the twisted cohomology groups where the coefficients are paired using an Ad–invariant inner product on the Lie algebra $\mathfrak{sl}_2\mathbb{R}$.

From the identification

$$T_\rho\mathcal{T}(S_g) \xrightarrow{\sim} H^1(S, \operatorname{Ad}\rho)$$

we obtain a bilinear form on $T_\rho\mathcal{T}(S_g)$

$$\omega : H^1(S, \operatorname{Ad}\rho) \times H^1(S, \operatorname{Ad}\rho) \xrightarrow{\smile} H^2(S, \mathbb{R}) \xrightarrow{\int} \mathbb{R}$$

This form is antisymmetric and nondegenerate. It extends the Weil-Petersson symplectic form to the entire (smooth locus of the) $\operatorname{PSL}_2\mathbb{R}$ character variety. Goldman's construction generalizes to any Lie group admitting a nondegenerate Ad–invariant bilinear form on its Lie algebra. See [1] or [3] for more details on the above construction (the second reference also contains details for the general construction).

Wolpert later gave a third description of the Weil-Petersson form as

$$\omega_{WP} = \sum d\ell_{\alpha_j} \wedge d\tau_{\alpha_j}$$

in Fenchel-Nielsen coordinates discussed above. [5], [6]

3 Twist Flows

3.1 Earthquakes

We now describe the earthquake flow on Teichmüller space which will depend on a fixed simple closed curve $\alpha \subset S_g$ (or more generally on a measured geodesic lamination). We do so by describing the flow lines. That is given a point $(X_0, \phi_0) \in \mathcal{T}(S_g)$, we construct a new point (X_t, ϕ_t) obtained by applying the earthquake flow for time t.

First we describe how to obtain the hyperbolic structure X_t. Let $(X_0|\alpha)$ denote the hyperbolic surface X_0 split along α. It is a bordered hyperbolic surface with geodesic boundary, $\partial(X_0|\alpha) = \{\alpha_0^+, \alpha_0^-\}$, together with a gluing isometry $\iota_\alpha : \alpha_0^+ \to \alpha_0^-$ such that $(X_0|\alpha)/\iota_\alpha \cong X_0$. We change the hyperbolic structure by changing the gluing. Consider the family of isometries $\theta_t : \alpha_0^+ \to \alpha_0^-$ determined by $\theta_0 = \iota_\alpha$ and the property that for all $x \in \alpha_0^+$, the path $t \mapsto \theta_t(x)$ is a unit speed trajectory in α_0^-. Define X_t to be the hyperbolic surface obtained by identifying the geodesic boundaries of $(X_0|\alpha)$ via θ_t, that is $X_t := (X_0|\alpha)/\theta_t$.

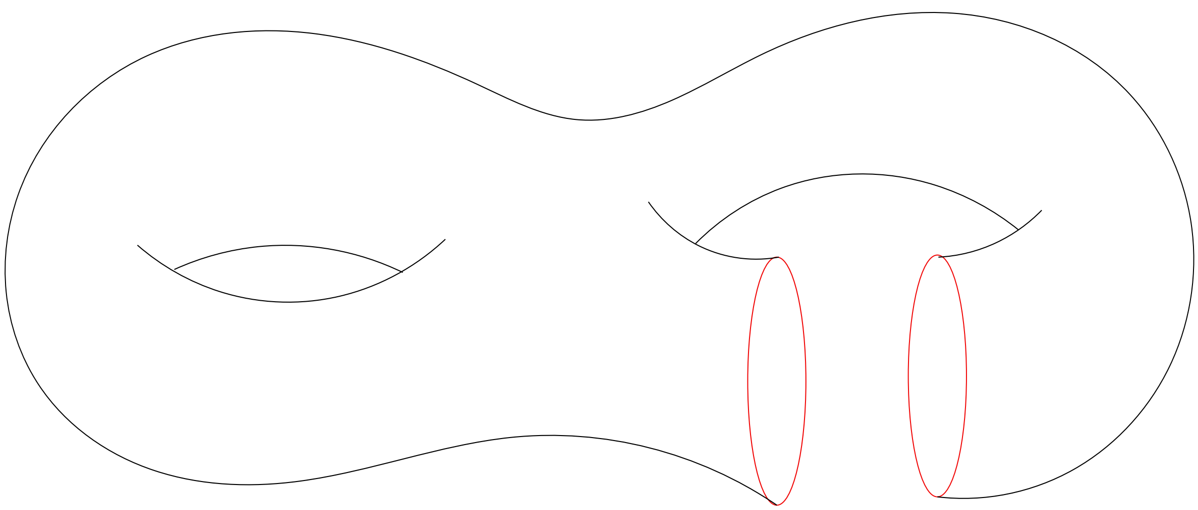

Figure 3: A surface split along a curve α

Now it remains to describe the marking. Let γ be a loop intersecting α (transversely). The curve split along α, $(\gamma|\alpha) \subset (X_0|\alpha)$, is a disjoint union of arcs with endpoints on α_0^+ and α_0^- such that we recover γ from the gluing ι_α. In particular, under the gluing θ_t, $(\gamma|\alpha)/\theta_t$ is no longer a loop. For each endpoint $x \in \alpha_0^+$ we insert to our curve $(\gamma|\alpha)$ the curve $s \mapsto \theta_{st}(x)$ for $0 \leq s \leq 1$. In the quotient X_t, we obtain a closed curve γ_t. This produces an isomorphism

$$\pi_1(X_0) \to \pi_1(X_t)$$
$$\gamma \mapsto \gamma_t$$

from which we recover a homotopy equivalence $h_t : X_0 \to X_t$. The marking is then just $\phi_t = h_t \circ \phi_0$.

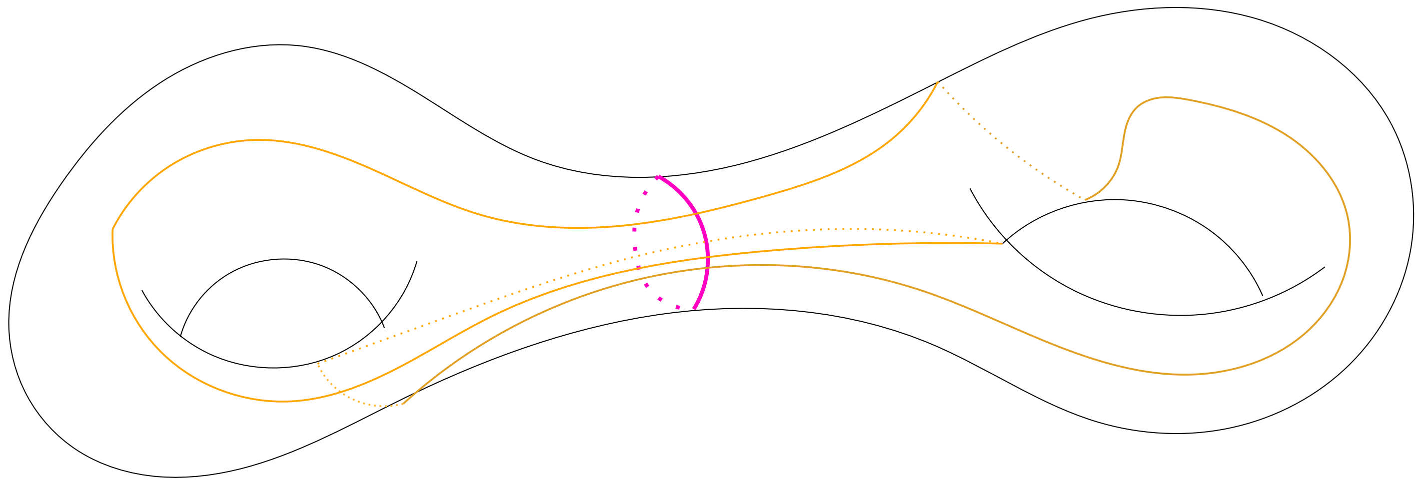

Figure 4: To illustrate the point that a curve split along α (in magenta) is a disjoint union of arcs, note curves on surfaces can get very complicated (pictured in orange).

3.2 Generalized twist flows

Noting the earthquake flow about α doesn't affect curves not intersecting α, Goldman defined a generalized twist flow as one occuring "upstairs" on $\mathrm{Hom}(\pi_1, G)$ which descends to a trivial deformation "downstairs" on the character variety.

Definition 3. *A* generalized twist flow *about a simple closed curve α is a flow $\{\eta_t\}_{t\in\mathbb{R}}$ on $\mathrm{Hom}(\pi_1, \mathrm{PSL}_2\,\mathbb{R})$ such that for all $\phi \in \mathrm{Hom}(\pi_1, \mathrm{PSL}_2\,\mathbb{R})$ and any component C of $(S|\alpha)$ there is a path $g_t = g_t(C)$ in $\mathrm{PSL}_2\,\mathbb{R}$ with $\eta_t\phi(\gamma) = g_t\phi(\gamma)g_t^{-1}$ for all $\gamma \in \pi_1(C) \subset \pi_1(S_g)$.*

The descriptions here for earthquakes and generalized twists follow Goldman [4].

4 Earthquakes are Hamiltonian

Theorem 2. *Let $\alpha \subset S_g$ be a simple closed curve and let $f : \mathrm{PSL}_2\,\mathbb{R} \to \mathbb{R}$ be conjugation invariant. Define $\widetilde{f}_\alpha : \mathrm{Hom}(\pi_1, \mathrm{PSL}_2\,\mathbb{R}) \to \mathbb{R}$ by $\widetilde{f}_\alpha(\phi) = (f \circ \phi)(\alpha)$. This descends to a function $f_\alpha : \mathrm{Hom}(\pi_1, \mathrm{PSL}_2\,\mathbb{R})/\,\mathrm{PSL}_2\,\mathbb{R}$. Then the Hamiltonian flow determined by f_α is covered by a generalized twist flow $\{\eta_t\}_{t\in\mathbb{R}}$ about α on $\mathrm{Hom}(\pi_1, \mathrm{PSL}_2\,\mathbb{R})$.*

As a corollary, Goldman recovers Wolpert's duality (see [4], [7])

Theorem 3. *The earthquake flow about a simple closed curve α is Hamiltonian with respect to the Weil-Petersson form and the length function, ℓ_α, determined by α*

Note that any curve $\alpha \in \pi_1(S_g)$ determines a length function (i.e. we do not require α to be

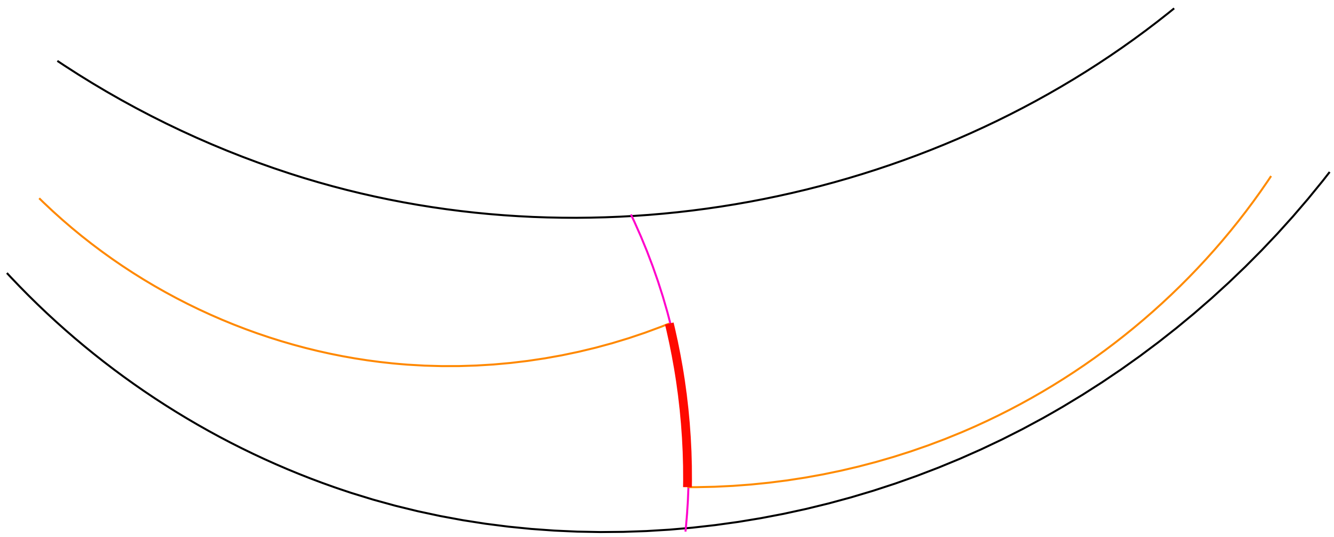

Figure 5: In magenta is the curve α we are twisting about. In orange is γ some curve intersecting α and in red is the segment we attach to close the loop.

in a pants decomposition as discussed above for the definition of ℓ_α to make sense).

Goldman proves this result of Wolpert by first constructing a generalized twist flow covering the earthquake flow about α. Using this and an explicit formulas he obtains in proving theorem 2, he concludes this generalized twist flow also covers the Hamiltonian flow associated to $f_\alpha = \ell_\alpha$ and hence conclude that the two flows coincide.

References

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