Symplectic Fibration and Connection

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1 Overview

This talk will give the main idea of proving Thurston's theorem which concerns the construction of a symplectic form ω on a manifold M that is compatible with a given compact symplectic fibration $(\pi : M \longrightarrow B)$ under some suitable conditions.

Notice that such ω need not always exist. For example, consider the Hopf bundle $(S^3 \times S^1, \pi, S^2)$. The Hopf bundle is a compact symplectic fibration and the fibre is just torus $S^1 \times S^1$ with standard symplectic form. However the $S^3 \times S^1$ does not admit a symplectic structure.

We will begin with some basic set up and some necessary definitions.

2 Basic set up and definitions

Let F be a smooth manifold. A **Fibration** with fibre F over a smooth manifold B consists of a smooth map $\pi : M \longrightarrow B$ such that the following conditions hold:

1. M is a smooth manifold.
2. π is surjective mapping.
3. For every point $p \in B$, there exists an open neighborhood $U \subset B$ and a diffeomorphism $\phi : \pi^{-1}(U) \longrightarrow U \times F$ such that $\pi|_U = pr \circ \phi$ where the map $pr : U \times F \longrightarrow U$ is the projection.

Let B be a connected smooth manifold equipped with an open cover $\{U_\alpha\}_\alpha$ and $\pi : M \longrightarrow B$ be a fibration with fibre F. For every α, there exists a diffeomorphisms $\phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times F$ such that the following diagram commutes.

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\phi_\alpha} & U_\alpha \times F \\ & \searrow^{\pi} & \downarrow{\scriptstyle pr} \\ & & U_\alpha \end{array}$$

We denote by $F_b = \pi^{-1}(b)$ the fibre over $b \in B$ and by $\phi_\alpha(b) : F_b \longrightarrow F$ the restriction of ϕ_α to F_b followed by the projection onto F.
The **transition map** $\phi_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow$Diff(F) is defined by

$$\phi_{\beta\alpha} = \phi_\beta(b) \circ \phi_\alpha^{-1}(b).$$

Very often, a fibration will carry some extra structure called a **structure group**. Let G be a subgroup of Diff(F). G is a **structure group** of $(\pi, \{\phi_\alpha\})$ if transition functions $\phi_{\beta\alpha}$ all take values in G.

Definition 2.1. Symplectic Fibration: Let $\pi : M \longrightarrow B$ be a locally trivial fibration with fiber (F, ω_F) which is a symplectic manifold. π is symplectic fibration if each fibre F_b carries a symplectic structure $\sigma_b \in \Omega^2(F_b)$ such that $\exists$ $\{\phi_\alpha\}$ such that

$$\sigma_b = \phi_\alpha^*(b)\sigma$$

for all $b \in U_\alpha$.

Once we have definition of symplectic fibration, we are ready to define the compatible symplectic form.

Definition 2.2. Compatible Symplectic form Let $\pi : M \longrightarrow B$ be a symplectic fibration with fibre (F, ω_F). A symplectic form $\omega \in \Omega^2(M)$ is called compatible with the symplectic fibration π if, $\omega|_{F_b} = \sigma_b$ for every $b \in B$.

Theorem 2.1. Let $\pi : M \longrightarrow B$ be a locally trivial fibration with connected base B, closed manifold M and $\omega \in \Omega^2(M)$ be a symplectic form such that the fibres are all symplectic submanifold of M. Then $\pi : M \longrightarrow B$ admits the structure of a symplectic fibration which is compatible with ω.

Proof. sketch of proof:
(1) Stokes' theorem implies that the symplectic forms $\sigma_b = {\iota_b}^*\omega \in \Omega^2(F_b)$ all represent the same cohomology class in $H^2(F)$ under map $\phi_\alpha(b)$ where $\iota_b : F_b \longrightarrow M$.
(2) Use Moser's stability theorem to show that the fibres (F_b, σ_b) are all symplectomorphic to a standard fibre (F, σ). Deduce that the structure group of $(\pi, \{\phi_\alpha\})$ can be reduced to $Symp(F, \sigma)$.
Remark. The detailed explanation is on *Introduction to Symplectic Topology* by D.Mcduff and D.Salamon page 253.

□

3 Thurston Theorem

Theorem 3.1. Let $\pi : (M, \omega) \longrightarrow (B, \beta)$ be compact symplectic fibration with symplectic fibre (F, σ) and connected symplectic base (B, β). Define $F_b = \pi^{-1}(b)$ where $b \in B$. Let σ_b be the symplectic form on F_b.
Let $a \in H^2(M)$ be a cohomology class such that for every $b \in B$,

$$\iota_b^*(a) = [\sigma_b]$$

where

$$\iota_b : F_b \longrightarrow M$$

is the inclusion map.

Then for every sufficiently large positive real number k, there exists a symplectic form $\omega_k \in \Omega^2(M)$ which is compatible with the fibration π and represents the class $a + k[\pi(\beta)^*]$.

Proof. Let $\{\phi_\alpha\}$ as in **Definition 2.1**. Let open sets $U_\alpha \subset B$ in the cover are chosen to be contractible. Let $\tau_0 \in \Omega^2(M)$ be any closed 2-form which represents the class $a \in H^2(M)$. For any α, let $\sigma_\alpha \in \Omega^2(U_\alpha \times F)$ be the 2-form obtained from (F, σ) via pullback map: $U_\alpha \times F \longrightarrow F$. First, we want to show the 2-forms

$$\phi_\alpha{}^* \sigma_\alpha - \tau_o \in \Omega^2(\pi^{-1}(U_\alpha))$$

are exact.
For fixed α and fixed $b \in U_\alpha$, let

$$\iota : F \longrightarrow U_\alpha \times F$$
$$x \mapsto (b, x).$$

Then

$$Pr_2 \circ \iota = Id_F$$

where

$$Pr_2 : U_\alpha \times F \longrightarrow F$$
$$(b, x) \mapsto x.$$

For the following commuting diagram:

$$\begin{array}{ccccc} H^2(F) & \xrightarrow{Pr_2{}^*} & H^2(U_\alpha \times F) & \xrightarrow{(\iota^* : \cong)} & H^2(F) \\ & & \big\downarrow {\scriptstyle (\phi_\alpha{}^* : \cong)} & & \big\downarrow {\scriptstyle (\phi_\alpha{}^*(b) : \cong)} \\ & & H^2(\pi^{-1}(U_\alpha)) & \xrightarrow{(\iota_b{}^* : \cong)} & H^2(F_b) \end{array}$$

$$\iota^* \circ Pr_2{}^* = (Pr_2 \circ \iota)^* = Id_{H^2(F)}. \tag{1}$$

It follows that

$$\iota^*[\sigma_\alpha] = [\sigma]$$

and therefore

$$\iota_b^*[\phi_\alpha{}^* \sigma_\alpha] = [\sigma_b]. \tag{2}$$

Now we need to check the map ι^*. Notice that ι^* is a linear map and (1) tells us this map should be surjective. Recall that U_α is chosen to be contractible.
Therefore we have:

$$H^2(U_\alpha \times F) \cong \mathbb{R} \otimes H^2(F) \cong H^2(F)$$

This implies that

$$dim(H^2(U_\alpha \times F)) = dim(H^2(F)) \tag{3}$$

By compactness of fibre F, we notice that $dim(H^2(F)) < \infty$. Using (3) together will rank-nullity theorem, this implies that ι^* is also an injective map. As ϕ_α^* and $\phi_\alpha^*(b)$ are isomorphisms, the same holds for ι_b^*. Finally, (2) together with the assumption in the theorem $\iota_b^*(a) = [\sigma_b]$ implies:

$$[{\phi_\alpha}^* \sigma_\alpha] = a \in H^2(\pi^{-1}(U_\alpha))$$

and therefore:

$$ {\phi_\alpha}^* \sigma_\alpha - \tau_o \in \Omega^2(\pi^{-1}(U_\alpha))$$

are exact. By exactness, we know there exist a collection of 1-forms $\lambda_\alpha \in \Omega^1(\pi^{-1}(U_\alpha))$ such that ${\phi_\alpha}^*\sigma_\alpha - \tau_o = d\lambda_\alpha$.
Second, we define a 2-form $\tau := \tau_o + \Sigma_\alpha d((\rho_\alpha \circ \pi)\lambda_\alpha \in \Omega^2(M)$ where $\{\rho_\alpha\}_\alpha$ a partition of unity subordinate to the cover $\{U_\alpha\}_\alpha$ and we want to check if τ is closed,represents the cohomology class $a \in H^2(M)$ and restricts to the form σ_b on each fibre.
For closedness, just notice that $d\tau = d\tau_o + d\Sigma_\alpha d((\rho_\alpha \circ \pi)\lambda_\alpha) = 0$.
Therefore we have $[\tau] = a$.

$$\begin{aligned}
{\iota_b}^*\tau &= {\iota_b}^*(\tau_o + \Sigma_\alpha d((\rho_\alpha \circ \pi)\lambda_\alpha))...① \\
&= {\iota_b}^*\tau_o + \Sigma_\alpha({\iota_b}^* d(\rho_\alpha \circ \pi) \wedge \lambda_\alpha + {\iota_b}^*((\rho_\alpha \circ \pi) \wedge d\lambda_\alpha))...② \\
&= {\iota_b}^*\tau_o + \Sigma_\alpha {\iota_b}^*((\rho_\alpha \circ \pi) d\lambda_\alpha)...③ \\
&= \Sigma_\alpha(\rho_\alpha \circ \pi){\iota_b}^*(\tau_o + d\lambda_\alpha)...④ \\
&= \Sigma_\alpha(\rho_\alpha \circ \pi){\iota_b}^*({\phi_\alpha}^*\sigma_\alpha)...⑤ \\
&= {\iota_b}^*({\phi_\alpha}^*\sigma_\alpha)...⑥ \\
&= \sigma_b...⑦
\end{aligned} \tag{4}$$

Remark. 1. We can ② to ③ of equation (4) by using ${\iota_b}^* d(\rho_\alpha \circ \pi) = 0$ since it vanishes on vectors tangent to the fibre.

2. Recall that ${\phi_\alpha}^*\sigma_\alpha - \tau_o = d\lambda_\alpha$ and $\phi_\alpha^*\sigma_\alpha \in H^2(\pi^{-1}(U_\alpha))$. Therefore we can reduce ④ to ⑤ line of equation (4).

Finally we can construct the 2-form for M :$\omega_k = \tau + k\pi^*\beta$ where k is a positive real number.
Thirdly, we want to explain why ω_k is closed and non-degenerate for sufficiently large k.
For closed, just notice that $d\omega_k = d\tau + kd\pi^*\beta = 0$.
For non-degeneracy:
For arbitrary $p \in M$, define $vert_p = ker(d\pi|_p) \subset T_pM$ and $hor_p = (vert_p)^\tau = \{u \in T_pM | \tau(u,v) =$

$0 \forall v \in vert_p\}$.
Notice that τ is non-degenerate on $vert_p$ as $\iota_b{}^*\tau = \sigma_b$. Therefore we can do the following splitting:

$$T_pM = vert_p \oplus hor_p$$

In order to finish the proof we need the following claims whose proofs are leave to readers.

1. $d\pi|_p : hor_p \longrightarrow T_{\pi(p)}B$ are linear isomorphisms.
2. $\exists k_0$ such that for all $k \geq k_0$ the ω_k are non-degenerate on the subbundle $hor \subset TM = \cup_{p\in M} T_pM$.
3. For every k, ω_k is non-degenerate on subbundle $vert = \cup_{p\in M} vert_p$ (notice that $\omega_k|_{vert} = \tau|_{vert}$).
4. The tangent bundle of M splits as $TM = vert \oplus hor$. Moreover $\forall k$, $\forall p \in M$, $\omega_k|_p(u, v) = 0$ for $u \in hor_p$ and $v \in vert_p$.

ω_k non-degenerate on M means $\forall \vec{u} \neq 0 \in TM$, $\exists \vec{v} \in TM$ such that $\omega_k(\vec{u}, \vec{v}) \neq 0$. For any non-zero $\vec{u} \neq 0 \in TM$, we can decompose it as $\vec{u} = \vec{h} + \vec{t}$ where $\vec{h} \in hor_p$ and $\vec{t} \in vert_p$. Note that $\vec{h}, \vec{t}$ can't be zero at same time since $\vec{u} \neq 0$.
Let k_0 be the number in the second claim. For every $\vec{h} \in hor_p$ and $\vec{t} \in vert_p$, we need to discuss the following cases:

1. Suppose $\vec{h} \neq 0$ and $\vec{t} = 0$. $\exists \vec{p} \in hor$ such that $\omega_k|_p(\vec{u}, \vec{p}) \neq 0$ since ω_k is non-degenerate on hor when $k > k_0$.
2. Suppose $\vec{h} = 0$ and $\vec{t} \neq 0$. $\exists$ $\vec{q} \in vert$ such that $\omega_k|_p(\vec{t}, \vec{q}) \neq 0$ by ω_k is non-degenerate on subbundle $vert$. Therefore, $\vec{q}$ is a vector in TM such that $\omega_k|_p(\vec{u}, \vec{q}) \neq 0$.
   Moreover, if any of $\omega_k|_p(\vec{h}, \vec{p})$ and $\omega_k|_p(\vec{t}, \vec{q})$ are negative, say $\omega_k|_p(\vec{t}, \vec{q}) < 0$, we can make it to be positive by multiplying -1 on $\vec{q}$. (ie, if $\omega_k|_p(\vec{t}, \vec{q}) < 0$, pick $\vec{q'} = -\vec{q}$ to replace $\vec{q}$. This will give you $\omega_k|_p(\vec{t}, \vec{q'}) > 0$.)
3. Suppose both $\vec{h}, \vec{t}$ are non zero vectors and let $k > k_0$. $\exists \vec{p} \in hor$ and $\exists \vec{q} \in vert$ such that $\omega_k|_p(\vec{h}, \vec{p}) > 0$ and $\omega_k|_p(\vec{t}, \vec{q}) > 0$ since ω_k is non-degenerate on subbundle $vert$ and hor. Therefore $\omega_k|_p(\vec{u}, \vec{p}+\vec{q}) = \omega_k|_p(\vec{u}, \vec{p}) + \omega_k|_p(\vec{u}, \vec{q}) = \omega_k|_p(\vec{h}, \vec{p}) + \omega_k|_p(\vec{h}, \vec{q}) + \omega_k|_p(\vec{t}, \vec{p}) + \omega_k|_p(\vec{t}, \vec{q}) = \omega_k|_p(\vec{h}, \vec{p}) + \omega_k|_p(\vec{t}, \vec{q}) \neq 0$.

Therefore, we know ω_k is non-degenerate on M if $k > k_0$.

Overall together with (1)-(4) will imply ω_k is a symplectic form on M which is compatible with the bundle and represents the cohomology class $a + k[\pi^*\beta]$ for sufficiently large k.

□

4

References

[1] Ana Cannas da Silva *Lectures on Symplectic Geometry* Springer,New York, first edition,2008

[2] D.Mcduff and D.Salamon *Introduction to Symplectic Topology* Oxford University Press Inc,New York, 2nd edition, 1998