1 Contact Structures

Given any manifold M a dimension k distribution is a subbundle of TM of dimension k. Contact geometry revolves around distributions of codimension 1 satisfying a particular nonintegrability condition.To state this condition note that locally any codimension 1 distribution, ξ, may be defined as the kernel of a nowhere vanishing 1-form α. This 1-form is unique up to multiplication by some nonvanishing function.

Definition 1. A contact structure on a manifold M is codimension 1 distribution ξ such that for any locally defining 1-form α, $d\alpha|_\xi$ is nondegenerate (yielding symplectic vector spaces pointwise). Any such α is then called a local contact form.

Proposition 1. A codimension 1 distribution ξ is a contact structure if and only if $\alpha \wedge (d\alpha)^n$ is nonvanishing for any local defining 1-form α. In particular if α is globally defined then we get a volume form.

Proof. If $d\alpha|_\xi$ is nondegenerate then at any point $p \in M$, by the normal form for antisymmetric bilinear forms,

$$T_pM = \xi_p \oplus \ker d\alpha_p = \ker \alpha_p \oplus \ker d\alpha_p$$

that is, there exists a basis $u, e_1, \dots, e_n, f_1, \dots, f_n$ for T_pM such that u spans $\ker d\alpha_p$ and $e_1, \dots, e_n, f_1, \dots, f_n$ form a symplectic basis for $\xi_p = \ker \alpha_p$. Thus

$$\alpha \wedge (d\alpha)^n(u, e_1, \dots, e_n, f_1, \dots, f_n) = \alpha(u)(d\alpha)^n(e_1, \dots, e_n, f_1, \dots f_n) \neq 0.$$

On the other hand if $\alpha \wedge (d\alpha)^n$ is nonvanishing then in particular $(d\alpha)^n$ must be nonvanishing on the kernel of α so $d\alpha|_\xi$ is nondegenerate.

□

Example 1. On $\mathbb{R}^{2n+1}$ with coordinates $(x_1, y_1, \dots, x_n, y_n, z)$ the 1-form $\alpha = dz + \sum_{j=1}^n x_j dy_j$ gives a contact structure. Indeed,

$$d\alpha = \sum_{j=1}^n dx_j \wedge dy_j$$

so

$$\alpha \wedge (d\alpha)^n = n! dz \wedge dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \neq 0.$$

There is a Darboux type theorem for contact manifolds showing that any contact form can be written locally in the above form:

Theorem 1. Given any contact manifold (M, ξ), near each point there are local coordinates $(x_1, y_1, \dots, x_n, y_n, z)$ on M such that

$$\alpha = dz + \sum_{j=1}^n x_j dy_j$$

is a local contact form for ξ.

This can be proved using the same strategy as Darboux's theorem for symplectic manifolds (see [1] for example) Alternatively, we will later see that any contact manifold admits an embedding into a symplectic manifold whereby symplectic form restricts to the derivative of the contact form. The above then becomes a corollary of Darboux's theorem.

2 Contact Dynamics

Proposition 2. Given any contact 1-form α there is a unique vector field R_α satisfying

1. $\iota_{R_\alpha} d\alpha = 0$,

2. $\alpha(R_\alpha) = 1$.

Moreover the flow by this R_α preserves α. Such a vector field is called the Reeb vector field associated to α.

Proof. The first condition means that R_α is a section of the line bundle $\ker d\alpha$ while the second condition amounts to a normalization of $R\alpha$.

Now let ϕ_t be the flow by R_α. Then by Cartan's magic formula

$$\frac{d}{dt}(\phi_t^*\alpha) = \phi_t^*(\mathcal{L}_{R_\alpha}\alpha) = \phi_t^*(d\iota_{R_\alpha}\alpha + \iota_{R_\alpha}d\alpha) = 0.$$

Thus, $\phi_t^*\alpha = \phi_0^*\alpha = \alpha$.

□

Example 2. In our model example $\mathbb{R}^{2n+1}$ with 1-form $\alpha = dz + \sum_{j=1}^n x_j dy_j$ we find that $\ker d\alpha = \operatorname{span}(\frac{\partial}{\partial z})$ so the Reeb vector field is $\frac{\partial}{\partial z}$.

3 Symplectization and Weinstein's Conjecture

We would like to put Weinstein's Conjecture into the proper framework of contact geometry. To do so we will show that any contact manifold can be taken as the level set of some Hamiltonian on a symplectic manifold. Given a contact manifold (M^{2n-1}, ξ), let α be some contact form giving ξ. Then consider the manifold $M \times \mathbb{R}$ and the closed 2-form $\omega = d(e^\tau \alpha)$ (where τ is the coordinate on $\mathbb{R}$). To see that this 2-form is nondegenerate, yielding a symplectic structure note that

$$\omega^n = (e^\tau(d\tau \wedge \alpha + d\alpha))^n = ne^{n\tau} d\tau \wedge \alpha \wedge (d\alpha)^{n-1} \neq 0$$

Furthermore if R_α is the Reeb vector field associated to α then

$$\iota_{R_\alpha}\omega = e^\tau\left(\iota_{R_\alpha}(d\tau \wedge \alpha) + \iota_{R_\alpha}d\alpha\right) = -e^\tau d\tau$$

so the restriction of the Hamiltonian vector field of the function $H(\tau) = e^\tau$ to any of its level sets gives the Reeb vector field R_α.

Now we recall that Weinstein's conjecture says that if a regular energy level set is of contact type then it contains a periodic trajectory of the Hamiltonian vector field. Since any contact manifold embeds as such a regular energy level set Weinstein's conjecture is really a statement about contact manifolds:

Conjecture 1 (Weinstein, [3])**.** Given a compact contact manifold M, with a global contact form α there exists a periodic orbit for the Reeb vector field R_α.

The importance of this conjecture is that it gives a sufficient condition under which the Hamiltonian has periodic orbits. In particular mechanical hypersurfaces of cotangent bundles, that is, level sets of a Hamiltonian $H = T + V$ where T is kinetic energy and V is potential energy are of contact type (see [2]).

References

[1] Hansjörg Geiges. *An introduction to contact topology*, volume 109. Cambridge University Press, 2008.

[2] Jan Bouwe van den Berg, Federica Pasquotto, Thomas O Rot, and RCAM Vandervorst. Closed characteristics on non-compact mechanical contact manifolds. *arXiv preprint arXiv:1303.6461*, 2013.

[3] Alan Weinstein. On the hypotheses of rabinowitz'periodic orbit theorems. *Journal of differential equations*, 33(3):353–358, 1979.