MATH1344 PRESENTATION TOPIC: INTEGRABLE SYSTEMS

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Let (M, ω, H) be a triple: symplectic manifold M of dimension $2n$ with symplectic form ω and a Hamiltonian function H.

Definition 1. An **integral (of motion)** is a function $f \in C^\infty(M)$ such that $\{f, H\} = 0$.

Note: H is an integral of motion.

Theorem 2 (Part of Arnold Liouville)**.** *Given (M, ω, H) and n integrals $f_1 = H, \dots, f_n$ such that $\{f_i, f_j\} = 0$ for all i, j. Let $c \in \mathbb{R}^n$ be a regular value of $f = (f_1, \dots, f_n)$. Suppose $M_c := f^{-1}(c)$ is compact and connected. Then*

(1) *M_c is a Lagrangian submanifold of M.*
(2) *M_c is diffeomorphic to the torus*
$$T^n = (\mathbb{R}/2\pi\mathbb{Z})^n = \{(\phi_1, \dots, \phi_n) \pmod{2\pi}\}$$
(3) *There exist angular coordinates $\phi = (\phi_1, \dots, \phi_n)$ such that in the Hamiltonian flow for $H = f_1$,*
$$\frac{d\phi}{dt} = h$$
on M_c, where $h \in \mathbb{R}^n$ is a constant.

In the following we give an outline of the proof (c.f. section 49,50 of V.I. Arnold).

Outline of Proof. Here we leave a few lemma unproved.

c is a regular value of f means for any $p \in M_c, df|_p = (df_1|_p, \dots, df_n|_p) : T_pM \to \mathbb{R}^n$ is surjective. Let W be the image of $df|_p$, so $W \subset \mathbb{R}^n$ is a linear subspace, and $W^\perp$ has complementary dimension. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$,
$$\begin{aligned}\sum_{i=1}^n \alpha_i df_i|_p = 0 &\Leftrightarrow \sum_{i=1}^n \alpha_i df_i|_p(v) = 0 \quad \forall \quad v \in T_pM \\ &\Leftrightarrow \sum_{i=1}^n \alpha_i w_i = 0 \quad \forall \quad w = (w_1, \dots, w_n) \in W \\ &\Leftrightarrow \alpha \in W^\perp\end{aligned}$$
Thus c is regular means $W = \mathbb{R}^n$, which is equivalent to $W^\perp = \{0\}$. This implies $\alpha = 0$, i.e. $df_i|_p$ are linearly independent.

The implicit function theorem implies that M_c is an n-dimensional submanifold of M. To conclude that M_c is Lagrangian, it remains to show that $\omega = 0$ on TM_c. Let $X_{f_i} : M_c \to TM_c$ be the vector fields associated with f_i, i.e. $\iota_{X_{f_i}}\omega = df_i$.

X_{f_i} are tangent to M_c: Denote $c = (c_1, \dots, c_n)$. Then $M_c = f_1^{-1}(c_1) \cap \dots \cap f_n^{-1}(c_n)$. It suffices to show that X_{f_i} is tangent to $f_j^{-1}(c_j)$ for all i, j. Equivalently, show that f_j is constant along integral curves of X_{f_i}, which we know is the condition $\{f_i, f_j\} = 0$.

Since df_i are independent at every point p of M_c, X_{f_i} generate TM_c. Then $\omega(X_{f_i}, X_{f_j}) = \{f_i, f_j\} = 0$, implying $w = 0$ on TM_c. This proves (1).

Let $g_i^t, i = 1, \dots, n$ be the flows of M_c corresponding to the n commuting vector fields X_{f_i}, then g_i^s and g_j^t commute, i.e. $g_i^s g_j^t x = g_j^t g_i^s x$. Define an action g of the commutative group $\{\xi \in \mathbb{R}^n\}$ on $M_c, g^\xi : M_c \to M_c$, by $g^\xi = g_1^{\xi_1} \dots g_n^{\xi_n}$ where $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$.

Fix $x_0 \in M_c$. Then we get a continuous map $g : \mathbb{R}^n \to M_c$ by $g(\xi) = g^\xi x_0$.

Lemma 3. *There are neighborhoods V of $0 \in \mathbb{R}^n$ and U of $x \in M_c$ such that the map $g : V \to U$ by $\xi \mapsto g^\xi x$ is a diffeomorphism.*

We claim that $g : \mathbb{R}^n \to M_c$ is onto: Let $y \in M_c$ and $\gamma : I \to M_c$ be a path from x_0 to y. The image $\gamma(I)$ is compact and has an open cover consisting of $\{U_x\}_{x \in \gamma(I)}$ where each U_x is taken from the previous lemma; hence $\gamma(I)$ is in fact covered by finitely many neighborhoods $U(x_0), \dots, U(x_m = y)$ as in the previous lemma.

Furthermore, we may assume that $x_{i+1} \in U(x_i) \cap U(x_{i+1})$ for all $0 \le i < m$: by subdividing components of $\gamma(U(x_i))$, we may assume that their preimages are open intervals in I. If $\gamma(U(x_i)) \subset \gamma(U(x_j))$, we may remove $U(x_i)$ and relabel accordingly. Now order the indices i according to $t_i^+ := \sup\{t \in I : \gamma(t) \in U(x_i)\}$. Note that $t_i^+ < t_{i+1}^+$ with strict inequality, since otherwise we have $\gamma(U(x_i)) \subset \gamma(U(x_{i+1}))$ or the reverse inclusion according to whether $t_i^- \ge t_{i+1}^-$, where $t_i^- := \inf\{t \in I : \gamma(t) \in U(x_i)\}$.

Starting with $i = 0, \gamma(t_i) \notin U(x_i)$ by assumption (otherwise $U(x_0), \dots, U(x_i)$ already cover $\gamma(I)$ and $i = m$). Then there is $j \ge i + 1$ such that $\gamma(t_i) \in U(x_j)$. We can pick $j = i + 1$ since otherwise $t_j^- \le t_{i+1}^-$, implying $\gamma(U(x_{i+1})) \subset \gamma(U(x_j))$. Pick $\epsilon_i > 0$ such that $x'_{i+1} := \gamma(t_i - \epsilon_i) \in U(x_i) \cap U(x_{i+1})$.

Observe that $U(x_{i+1})$ is also a neighborhood of x'_{i+1} such that the previous lemma also holds. Relabel each x'_{i+1} by x_{i+1}, then the assertion is true.

Since g is onto in each $U(x_i) \cap U(x_{i+1})$, we can find $\eta_i \in \mathbb{R}^n$ such that $g^{\eta_i} x_i = x_{i+1}$. Then $g(\eta_0 + \dots + \eta_{m-1}) = g^{\eta_{m-1}} \dots g^{\eta_0} x_0 = g^{\eta_{m-1}} \dots g^{\eta_1} x_1 = \dots = x_m = y$ as required.

Let $\Gamma = \{\xi \in \mathbb{R}^n \mid g^\xi x_0 = x_0\}$ be the stabilizer of x_0 in $\mathbb{R}^n$.

Lemma 4. *Γ is a discrete subgroup, i.e. there is a neighborhood V of $0 \in \mathbb{R}^n$ such that $\Gamma \cap (\xi + V) = \{\xi\}$ for all $\xi \in \Gamma$.*

The following is a general fact about discrete subgroups of $\mathbb{R}^n$.

Lemma 5. *There are linearly independent vectors $e_1, \dots, e_k \in \Gamma$ with $0 \le k \le n$, such that $\Gamma = \left\{ \sum_{i=1}^k m_i e_i \mid m_i \in \mathbb{Z} \right\}$.*

Let $p : \mathbb{R}^{2n} \to T^k \times \mathbb{R}^{n-k}$ be defined by $p(\phi, y) = (\phi \pmod{2\pi}, y)$, which is an universal covering space map. Here $\phi_1, \dots, \phi_k, y_1, \dots, y_{n-k}$ are coordinates on $T^k \times \mathbb{R}^{n-k}$. The points $\beta_i = (0, \dots, 0, \phi_i = 2\pi, 0 \dots, 0, y = 0)$ map to 0 under p.

Let $e_i \in \Gamma$ be as in the lemma, and define $A : \{(\phi, y) \in \mathbb{R}^n\} \to \{\xi \in \mathbb{R}^n\}$ to be an isomorphism such that $\beta_i \mapsto e_i$.

$$\begin{array}{ccc} \mathbb{R}^n = \{(\phi, y)\} & \to^A & \mathbb{R}^n = \{\xi\} \\ \downarrow^p & & \downarrow^g \\ T^k \times \mathbb{R}^{n-k} = \{(\phi, y)\} & \to^{\tilde{A}} & M_c \end{array}$$

Lemma 6. *A descends to a diffeomorphism $\tilde{A} : T^k \times \mathbb{R}^{n-k} \to M_c$.*

Since M_c is compact, $n = k$, which proves (2).

The Hamiltonian flow is $\psi_t^H(x) = g_1^t x = g^{t(1,0,\dots,0)}x$. Let $\eta = (1, 0, \dots, 0) \in \mathbb{R}^n$.

Now Γ has rank n with basis $e_1, \dots, e_n$. Let $s_1, \dots, s_n$ be the standard basis of $\mathbb{R}^n$ and let $A : \{\phi \in \mathbb{R}^n\} \to \{t \in \mathbb{R}^n\}$ with $s_i \mapsto e_i$ be the isomorphism as constructed above.

Since g is onto, for all $x \in M_c$ there is $z \in \mathbb{R}^n$ such that $g(z) = x$. Then

$$\begin{aligned}\psi_t^H(x) &= g^{t\eta}x = g^{t\eta}g(z) = g^{t\eta}g^z x_0 = g^{t\eta+z}x_0 = g(t\eta + z)\\ &= g(AA^{-1}(t\eta + z)) = (g \circ A)(tA^{-1}\eta + A^{-1}z)\end{aligned}$$

By the above commutative diagram, the same formula holds with $g \circ A$ replaced by $p \circ \tilde{A}$. The diffeomorphism $\tilde{A}$ defines angular coordinates $\phi = (\phi_1, \dots, \phi_n)$ in $M_c \simeq T^n$, satisfying $\phi(t) = \phi(0) + ht$ with $h = A^{-1}\eta$ and $\phi(0) = A^{-1}z$. So $\frac{d\phi}{dt} = h$. This proves (3). $\square$

References

[1] Anna Cannas da Silva, *Lectures on Symplectic Geometry*, Series 1764 Lecture Notes in Mathematics, Springer-Verlag Berlin Heidelberg, 2008. eBook ISBN:978-3-540-45330-7

[2] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag New York Berlin Heidelberg, 2nd edition, 1989. ISBN:0-387-96890-3