POISSON GEOMETRY

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1. Introduction

This document is intended as a brief introduction to the study of Poisson manifolds, with a focus on their local structure. Following the first half of Weinstein's paper [3], we discuss the basic properties of Poisson manifolds, state the splitting theorem, and introduce the topic of linearization. Some of our exposition is also based on notes by Fernandes and Mărcuț [1], which provide a more detailed look at this material.

2. Basic definitions and properties

Definition. A *Poisson structure* on a manifold M is a Lie bracket $\{,\}$ on $C^\infty(M)$ that satisfies the Leibniz property

$$\{fg, h\} = f\{g, h\} + \{f, h\} g$$

for all $f, g, h \in C^\infty(M)$. The pair $(M, \{,\})$ is called a *Poisson manifold*.

It follows from the Leibniz property and antisymmetry that $\{,\}$ is a derivation in each argument. In particular, for every $h \in C^\infty(M)$, the operator

$$X_h := \{\cdot, h\}$$

is a vector field, called the *Hamiltonian vector field* generated by h. (Warning: some authors instead define $X_h = \{h, \cdot\}$.)

Lemma 1. *For all $f, g \in C^\infty(M)$,*

$$X_{\{f,g\}} = [X_g, X_f]$$

(i.e., the map $C^\infty(M) \to \mathfrak{X}(M), f \mapsto X_f$, is a Lie algebra antihomomorphism).

Proof. For any $h \in C^\infty(M)$,

$$\begin{aligned} X_{\{f,g\}}(h) &= \{h, \{f, g\}\} \\ &= -\{g, \{h, f\}\} - \{f, \{g, h\}\} && \text{(Jacobi identity)} \\ &= \{\{h, f\}, g\} - \{\{h, g\}, f\} && \text{(antisymmetry)} \\ &= X_g(X_f(h)) - X_f(X_g(h)) = [X_g, X_f](h). \end{aligned}$$

□

Given a Poisson structure $\{,\}$ on M, there is a corresponding bivector field $\pi \in \mathfrak{X}^2(M) := \Gamma(\bigwedge^2 TM)$ such that

$$\pi(df, dg) = \{f, g\}. \tag{1}$$

Viewing π as a map $T^*M \times T^*M \to \mathbb{R}$, we obtain (by contraction) a map $\pi^\sharp : T^*M \to TM$. Note that (according to our sign conventions) we have $\pi^\sharp(dh) = -X_h$ for all $h \in C^\infty(M)$.

Remark. Given an arbitrary bivector field π, the bracket defined by (1) may not satisfy the Jacobi identify. One can show that π defines a Poisson bracket if and only if the Schouten bracket $[\pi, \pi]$ is zero. In this case, we will also refer to π as a *Poisson structure*.

In local coordinates $(x_1, \ldots, x_n)$ on M, we can write

$$\{f, g\} = \sum_{i,j=1}^{n} \{x_i, x_j\} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$$

and

$$\pi = \sum_{i<j} \pi_{ij}(x) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \quad \text{where } \pi_{ij}(x) = \{x_i, x_j\}.$$

Thus the Poisson structure is completely determined by the components $\pi_{ij}(x) = \{x_i, x_j\}$.

Example 1 (Classical bracket)**.** Let $M = \mathbb{R}^{2n}$ with coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$. Then

$$\{f, g\} := \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right) \tag{2}$$

is a Poisson structure, corresponding to the bivector field $\pi := \sum_{i=1}^{n} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}$.

Example 2 (Symplectic manifolds)**.** On any symplectic manifold (M, ω) there is a Poisson bracket such that the corresponding X_f are the usual symplectic Hamiltonian vector fields (i.e., they satisfy $df = \iota_{X_f} \omega$). In local symplectic coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$, this bracket is just given by (2).

3. Symplectic foliation

Definition. The *rank* of a Poisson structure π at $x \in M$ is the rank of the (linear) map $\pi_x^\sharp : T_x^* M \to T_x M$. (In local coordinates, this is given by the rank of the matrix $(\pi_{ij}(x))$.)

Definition. A Poisson structure π on M is called *nondegenerate* (or *symplectic*) if its rank is equal to $\dim M$ everywhere.

Remark. Example 2 gives a bijective correspondence between symplectic forms ω and nondegenerate Poisson structures π, justifying the above nomenclature.

Theorem 1. *If a Poisson structure π has constant rank on M, then* $\operatorname{Image} \pi^\sharp$ *is an integrable distribution which gives rise to a foliation of M into symplectic leaves.*

Outline. Since π has constant rank, $\operatorname{Image} \pi^\sharp$ is a subbundle of TM, i.e., a distribution. By definition of $\pi^\sharp$, this distribution is spanned by Hamiltonian vector fields. Since the bracket of Hamiltonian vectors fields is again Hamiltonian (by Lemma 1), we see that $\operatorname{Image} \pi^\sharp$ is involutive, and therefore integrable (by the Frobenius theorem).

Showing that the leaves of the corresponding foliation are symplectic requires the technical notion of an induced Poisson structure; we will omit the details here. □

Remark. For a general Poisson structure, one obtains a *singular foliation* of M into symplectic leaves (i.e., the leaves may have different dimensions).

Example 3. Consider $\mathbb{R}^{2n+s}$ with coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n, c_1, \ldots, c_s)$, and with the Poisson bracket bracket given by (2). (Thus, functions of $(c_1, \ldots, c_s)$ have zero bracket with all of $C^\infty(M)$.) In this case, the symplectic leaves are the submanifolds on which all of the c_i are constant, i.e., they are copies of $\mathbb{R}^{2n}$. Clearly $\mathbb{R}^{2n+s}$ is a union of such submanifolds.

4. Splitting

The product of Poisson manifolds can be given a Poisson structure. Suppose M_1 and M_2 have Poisson structures specified in local coordinates by relations $\{x_i, x_j\} = \pi^1_{ij}(x)$ and $\{y_i, y_j\} = \pi^2_{ij}(y)$ respectively. If we additionally specify that $\{x_i, y_j\} = 0$, then these relations together define a Poisson structure on $M_1 \times M_2$.

Theorem 2 (Splitting)**.** *Let M be a Poisson manifold, and let $x \in M$. Then there exists a neighbourhood U of x and an isomorphism $\phi = \phi_S \times \phi_P : U \to S \times P$ where*

- *S is symplectic;*
- *P is Poisson, with rank 0 at $\phi_P(x)$.*

Furthermore, the factors S and P are unique up to local isomorphism.

Remark. One possible representative for the factor S is the symplectic leaf through x.

This theorem reduces the local study of Poisson manifolds to the case where the rank at a point is equal to 0. This case will be studied in further detail in Section 6.

5. Linear Poisson structures

Fix a finite-dimensional Lie algebra $\mathfrak{g}$. For any $f \in C^\infty(\mathfrak{g}^*)$ and $\mu \in \mathfrak{g}^*$, the differential $df|_\mu$ is an element of $\mathfrak{g}^{**} \cong \mathfrak{g}$. Making this identification, we can define a Poisson structure on $\mathfrak{g}^*$, called the *Lie-Poisson structure*, by

$$\{f, g\}(\mu) = \langle\, \mu, [df|_\mu, dg|_\mu]\,\rangle\,, \qquad f, g \in C^\infty(M)$$

(where $\langle\, \cdot, \cdot\,\rangle$ is the pairing of $\mathfrak{g}^*$ with $\mathfrak{g}$).

We will describe the Lie-Poisson structure in coordinates. Let $x_1, \ldots, x_r$ be a basis for $\mathfrak{g}$, with corresponding structure constants c^k_{ij} (so that $[x_i, x_j] = \sum_k c^k_{ij} x_k$). Abusing notation, we can also view the x_i as coordinates on $\mathfrak{g}^*$ (via the identification of $\mathfrak{g}^{**}$ with $\mathfrak{g}$). Then the components of the Lie-Poisson structure are just the linear functions

$$\{x_i, x_j\} = \sum_k c^k_{ij} x_k.$$

The following fact is worth mentioning, although we will not use it later.

Fact. The symplectic leaves in $\mathfrak{g}^*$ are the coadjoint orbits.

6. Linear approximation

Let M be a manifold with Poisson structure π. Suppose that π has rank 0 at a point $p \in M$. We will show that we can provide T_pM with a Lie-Poisson structure.

First note that

$$T^*_pM \cong \mathfrak{m}^2_p/\mathfrak{m}_p,$$

where $\mathfrak{m}_p \subseteq C^\infty(M)$ is the ideal of functions vanishing at p.

Claim. *$\mathfrak{m}^2_p$ is a Lie ideal of $\mathfrak{m}_p$.*

Proof. Let $f, g, h \in \mathfrak{m}_p$, and write

$$\{f, gh\} = \{f, g\}\, h + g\,\{f, h\}\,.$$

The brackets $\{f, g\}$ and $\{f, h\}$ belong to $\mathfrak{m}_p$ (i.e., vanish at p) since the rank of the Poisson structure at p is 0. Hence $\{f, gh\} \in \mathfrak{m}^2_p$. □

Therefore $\mathfrak{g}_p := T_p^*M$ is a Lie algebra, which means that $\mathfrak{g}_p^* = T_pM$ can be given the Lie-Poisson structure. We call this the *linear approximation* to the Poisson structure at p.

The linear approximation has a simple description in coordinates. Suppose $x_1, \ldots, x_r$ are coordinates on M which vanish at p. Since $\pi_{ij} = \{x_i, x_j\}$ vanishes at 0 for all i, j, we can write the Taylor expansion

$$\pi_{ij}(x) = \sum_k c_{ij}^k x_k + O(x^2), \quad \text{where } c_{ij}^k := \frac{\partial \pi_{ij}}{\partial x_k}(0).$$

Then the c_{ij}^k are the structure constants of $\mathfrak{g}_p$ (as the notation suggests), and the components of the linear approximation Poisson structure are just given by

$$\sum_k c_{ij}^k x_k$$

(i.e., by removing the higher order terms).

7. Linearization

The notion of linear approximation leads to the question of linearization: When is the linear approximation at a point isomorphic to the original Poisson structure? The following example shows that this is not always the case.

Example 4. One can define a nontrivial Poisson structure on $\mathbb{R}^3$ by

$$\begin{aligned} \{x_1, x_2\} &= |x|^2 x_3 \\ \{x_2, x_3\} &= |x|^2 x_1 \\ \{x_3, x_1\} &= |x|^2 x_2, \end{aligned}$$

but the linear approximation to this structure at the origin is trivial (i.e., the zero bracket).

Definition. A Lie algebra $\mathfrak{g}$ is said to be formally/analytically/C^∞ *nondegenerate* if any Poisson manifold whose linear approximation at a point p is isomorphic to $\mathfrak{g}^*$ is itself isomorphic to $\mathfrak{g}^*$ at p, via a formal/analytic/C^∞ local isomorphism.

The classification of nondegenerate Lie algebras is a difficult problem, but certain results are known. For instance, Weinstein proves (in Theorem 6.1 of [3]) that any semisimple Lie algebra is formally nondegenerate. A more recent survey of the linearization problem can be found in [2].

References

1. Rui Loja Fernandes and Ioan Marcut, *Lectures on Poisson geometry.*
2. Rui Loja Fernandes and Philippe Monnier, *Linearization of Poisson brackets*, Letters in Mathematical Physics **69** (2004), no. 1, 89–114.
3. Alan Weinstein, *The local structure of Poisson manifolds*, J. Differential Geom. **18** (1983), no. 3, 523–557.