1 Introduction

Take (M, ω) a symplectic manifold, G a compact Lie group with Hamiltonian action on M, and $J : M \to \mathfrak{g}^*$ the momentum map. Recall this means J is G-equivariant (with respect to the coadjoint action), and $d\langle J, \mu\rangle = \iota_{\mu_M}\omega$ for each $\mu \in \mathfrak{g}$. Note the orbits of G carry a unique differential structure as immersed submanifolds of M. Warming up, we begin with a standard fact (see, for example, [2], page 142).

Lemma 1. Set $\mathfrak{g}_p := \{\mu \in \mathfrak{g} \mid \mu_M(p) = 0\}$, and say $\mathcal{O}_p$ is the orbit through p. Then

$$\operatorname{ann}\operatorname{Im} dJ_p = \mathfrak{g}_p, \quad \ker dJ_p = (T_p\mathcal{O}_p)^{\omega_p},$$

where ann is the annihilator. If $J(p) = 0$, then $T_p\mathcal{O}_p$ is isotropic, so the immersed submanifold $\mathcal{O}_p$ is isotropic in M.

Proof. For any $u \in T_pM$, $\mu \in \mathfrak{g}$,

$$\begin{aligned}
\langle dJ_p u, \mu\rangle &= d\langle J, \mu\rangle_p u && \text{by the chain rule}\\
&= \iota_{\mu_M}\omega_p(u) && \text{By Hamilton's equation}\\
&= \omega_p(\mu_M(p), u),
\end{aligned}$$

so the left is zero for all u if and only if $\mu_M(p) = 0$, proving the first statement. The left side is zero for all μ if and only if $u \in (T_p\mathcal{O}_p)^{\omega_p}$ (using the fact the tangent space consists of all $\mu_M(p)$), proving the second statement. For the last statement, taking $u = \xi_M(p)$ for $\xi \in \mathfrak{g}$, and evaluating $dJ_p u$, we get

$$\begin{aligned}
dJ_p(\xi_M(p)) &= dJ_p\left(\frac{d}{dt}\Big|_{t=0} \exp(t\xi)\cdot p\right) && \text{by definition of } \exp\\
&= \frac{d}{dt}\Big|_{t=0} (J(\exp(t\xi)\cdot p)) && \text{by the chain rule}\\
&= \frac{d}{dt}\Big|_{t=0} (\exp(t\xi)\cdot J(p)) && \text{because } J \text{ is invariant.}
\end{aligned}$$

If $J(p) = 0$, it follows that $0 = \langle dJ_p(\xi_M(p)), \mu\rangle = \omega_p(\mu_M(p), \xi_M(p)) = 0$. Since this holds for any $\mu \in \mathfrak{g}$, we conclude $T_p\mathcal{O}_p \subseteq \ker dJ_p = (T_p\mathcal{O}_p)^{\omega_p}$. □

Suppose now that G acts freely on M. By Lemma 1, 0 is a regular value of J, so $J^{-1}(0)$ is an embedded submanifold of M. The group G still acts properly and freely on $J^{-1}(0)$, so

$$M_0 := J^{-1}(0)/G$$

has a manifold structure. In 1974, Marsden and Weinstein showed there is a unique symplectic form ω_0 on M_0 such that given

$$\begin{tikzcd} J^{-1}(0) \arrow[r, hook, "\iota"] \arrow[d, "\pi"] & M \\ M_0 = J^{-1}(0)/G & \end{tikzcd}$$

we have $\pi^*\omega_0 = \iota^*\omega$. We call (M_0, ω_0) Marsden-Weinstein reduced space. If the G-action is not necessarily free, and 0 is not a regular value, then $J^{-1}(0)$ need not be a submanifold of M, and similarly for the quotient $M_0 = J^{-1}(0)/G$. Regardless M_0 is a topological space. In 1991 Sjamaar and Lerman in "Stratified symplectic spaces and reduction" [1] showed M_0 is a *stratified symplectic space*. The remainder of this note is devoted stating Sjamaar and Lerman's result. Cushman and Sjamaar wrote a helpful summary of [1], appearing in [3].

2 Statement of the Result

Definition 1. A *decomposition* of a Hausdorff and paracompact topological space X is a partition $X = \bigsqcup_{i \in I} S_i$, where the collection $\{S_i\}$ is locally finite, each S_i is locally closed, each S_i carries a manifold structure, and

$$\text{(frontier condition)} \quad S_i \cap \bar{S}_j \neq \emptyset \iff S_i \subseteq \bar{S}_j.$$

We call a space X with a decomposition a decomposed space. The frontier condition gives a partial ordering of the *pieces* S_i by $S_i \leq S_j$ if $S_i \cap \bar{S}_j \neq \emptyset$ (and equality if and only if $i = j$). This partial ordering is used to define a stratification of X. This additional structure on a decomposed space is defined recursively, like that of a CW or simplicial complex:

Definition 2. A *stratification* of a decomposed space X is an assignment to each $x \in S$ an open neighbourhood U of x in X, an open ball B about x in S, a stratified space L, and a homeomorphism $\varphi : B \times CL \to U$ preserving the decomposed structure.[1]

A decomposed space with a stratification is a stratified space. This is well-defined because the cone over a decomposed space increases the *depth* of a decomposed space by 1, where

$$\text{depth } X := \sup_{i \in I} \sup\{n \mid S_i < S_{i_1} < \cdots < S_{i_n}\}.$$

A relevant example comes from a compact Lie group G acting on a manifold M. For each pair of stabilizer subgroups $H, K \leq G$, declare $H \sim K$ if K and H are conjugate. In the set of stabilizer subgroups, denote the classes of this relation by (H), and define the *orbit type strata*

[1]Here CL is the infinte cone over L, obtained from $L \times [0, \infty)$ by identifying $L \times \{0\}$ with a point.

$$M_{(H)} := \{p \in M \mid (G_p) = (H)\}.$$

The connected components $M_{(H),i}$ of each $M_{(H)}$ are manifolds, and moreover the partition $M = \bigsqcup_{(H),i} M_{(H),i}$ is a decomposition of M. The corresponding partition of M/G by the $M_{(H),i}/G$ is decomposition of M/G. Both decomposed spaces are in fact stratified. We now state Sjamaar and Lerman's main result, Theorem 2.1 in [1].

Theorem 1. Take (M, ω) a symplectic manifold, equipped with a Hamiltonian action by the compact Lie group G with momentum map $J : M \to \mathfrak{g}^*$. For each stabilizer group $H \leq G$, the intersection $M_{(H),i} \cap J^{-1}(0)$ is a manifold (Lemma 1), and

$$(M_0)_{(H),i} = (M_{(H),i} \cap J^{-1}(0))/G$$

has a symplectic structure $(\omega_0)_{(H),i}$ such that given

$$\begin{array}{ccc} M_{(H),i} \cap J^{-1}(0) & \overset{\iota}{\hookrightarrow} & M \\ \downarrow{\scriptstyle \pi} & & \\ (M_0)_{(H),i} & & \end{array}$$

we have $\pi^*(\omega_0)_{(H),i} = \iota^*\omega$. Moreover, the $(M_0)_{(H),i}$ stratify M_0 with symplectic pieces.

3 Poisson structure on reduced space

We give another view of M_0. In a Poisson manifold P, a symplectic leaf about $x \in P$ is collection of all other $y \in P$ that can be joined to x by trajectories of a finite number of Hamiltonian vector fields. In 1983 Weinstein proved each symplectic leaf is a weakly embedded submanifold of P which carries a symplectic form.

In our case, M_0 is not necessarily a manifold, let alone Poisson. Regardless, it is stratified. Sjamaar and Lerman define a *smooth structure* on a stratified space X to be an algebra of continuous functions on X such that each member restricts to a smooth function on the pieces of X. If the pieces of X are symplectic manifolds, and there is a smooth structure $C^\infty(X)$ on X with a Poisson bracket $\{\cdot,\cdot\}$ making the embeddings $S \hookrightarrow X$ Poisson, we say $(X, \{\cdot,\cdot\})$ is a *stratified symplectic space*:

In the case $X = M_0$, Sjamaar and Lerman give a smooth structure $(C^\infty(M_0), \{\cdot,\cdot\})$ making M_0 a stratified symplectic space. Moreover, there is a notion of symplectic leaves in $(M, \{\cdot,\cdot\})$, and these are exactly the pieces of M_0 (i.e. the $M_{(H),i}$). Details follow.

Definition 3.
- A continuous function on M_0 is *smooth* if there is a smooth G-invariant function F on M such that $F|_{J^{-1}(0)} = f \circ \pi$. Denote this smooth structure by $C^\infty(M_0)$. It is a combination of the notion of Whitney smooth, and of smooth functions on an orbit space.

- A bracket of smooth functions $f, h \in C^\infty(M_0)$ is given by

$$\{f, h\} := \{F, H\}_M,$$

  where $F, H : M \to \mathbb{R}$ are G-invariant smooth functions witnessing the smoothness of f, h, and $\{\cdot, \cdot\}_M$ is the bracket induced by ω.

It takes work to show this bracket is well-defined. If 0 is a regular value of J, this Poisson structure coincides with the Marsden-Weinstein one induced by ω_0. Now we define the notion of a Hamiltonian vector field of $h \in C^\infty(M_0)$, and its trajectory.

Definition 4.
- The *Hamiltonian derivation* $\operatorname{ad} h$ of $h \in C^\infty(M_0)$ operates on smooth functions

$$\operatorname{ad} h \cdot f := \{h, f\}.$$

- A *trajectory* of $\operatorname{ad} h$ through a point in M_0 is a smooth curve[2] $\gamma : [0, 1] \to M_0$ starting at the point such that for all smooth $f : M_0 \to \mathbb{R}$,

$$\frac{d}{dt} f \circ \gamma(t) = -\operatorname{ad} h \cdot f(\gamma(t)).$$

Integral curves always exist in some neighbourhood, and are unique. We say the *leaves* of $(M_0, \{\cdot, \cdot\})$ are the equivalence classes obtained from declaring two points related if they can be joined by a finite number of trajectories of Hamiltonian derivations. Sjamaar and Lerman showed (page 19 of [1])

Theorem 2. The $(M_0)_{(H),i}$ are exactly the leaves of $(M_0, \{\cdot, \cdot\})$.

4 An example

We follow Example 2.4 in [3], to show reduced space may indeed fail to be a manifold. Set $M := \mathbb{C}^4$, with S^1 action

$$e^{i\theta} \cdot (z_1, z_2, z_3, z_4) := (e^{i\theta} z_1, e^{i\theta} z_2, e^{-i\theta} z_3, e^{-i\theta} z_4).$$

This action is Hamiltonian, with momentum map

[2]a continuous map $\gamma : [0, 1] \to M_0$ such that its pullback by any smooth function is smooth

$$J(z_1, z_2, z_3, z_4) := \frac{1}{2}(|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2).$$

Note 0 is not a regular value of J. As a set,

$$J^{-1}(0) = \{|z_1|^2 + |z_2|^2 = |z_3|^2 + |z_4|^2\}.$$

Consider $S^7 = \{|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2\} \subset \mathbb{C}^4$. Both S^7 and $J^{-1}(0)$ are S^1-invariant, and their intersection is the 6-dimensional S^1-invariant submanifold (of $\mathbb{C}^4$)

$$J^{-1}(0) \cap S^7 = S^3 \times S^3 \subset \mathbb{C}^2 \times \mathbb{C}^2, \quad \text{where each } S^3 \text{ has radius } \frac{1}{2}.$$

This is a deformation retract of $J^{-1}(0) \smallsetminus \{0\}$, which is $J^{-1}(0) \cap M_{\{1\}}$. Evidently

$$\begin{aligned} &S^1 \text{ acts on the first copy of } S^2 \text{ by } e^{i\theta} \cdot (z_1, z_2) := (e^{i\theta} z_1, e^{i\theta} z_2) \\ &S^1 \text{ acts on the second copy of } S^2 \text{ by } e^{i\theta} \cdot (z_3, z_4) := (e^{-i\theta} z_3, e^{-i\theta} z_4). \end{aligned}$$

Denote the quotient of $S^3 \times S^3$ by the S^1 action as $S^3 \times_{S^1} S^3$. Observe reduced space $(\mathbb{C}^4)_0$ is homotopy equivalent to the cone over $S^3 \times_{S^1} S^3$, i.e.

$$(\mathbb{C}^4)_0 \sim C(S^3 \times_{S^1} S^3).$$

The 3rd degree rational local homology of the cone at its vertex is $\mathbb{Q}$, which is impossible for a 6-dimensional manifold. Therefore reduced space is not a manifold (in fact, it is not an orbifold).

References

[1] R. Sjamaar and E. Lerman, *Stratified symplectic spaces and reduction*, Annals of Mathematics **134** (1991), 375-422.

[2] A. Cannas da Silva, *Lectures on Symplectic Geometry*, 1st ed., Lecture Notes in Mathematics, vol. 1764, Springer-Verlag, Berlin, 2008.

[3] R. Cushman and R. Sjamaar, *On singular reduction of Hamiltonian spaces*, Colloque de géométrie symplectique et physique mathematique (Aix-en-Provence, France, 1990), Progress in Mathematics, vol. 99, Birkhäuser Boston, Boston, 1991, pp. 114-128.