# MAT157Y TERM TEST 4, MARCH 2018

Please answer all the questions. You can answer the questions in any order, but please use separate pages for different questions, and mark the question number at the top corner of the page.

The questions have equal weight. Clarity counts. A response "I don't know" and nothing more, to any question or section of a question, will give you 20% of the points for that question or section. You have 100 minutes. No aids allowed.

(1) For each of the following statements, determine if it is true or false, and explain in one or two sentences.

(a) "If $f$ is an infinitely differentiable function, and $P_n$ is the Taylor polynomial of order $n$ for $f$ at a point $a$, then $\lim_{n\to\infty} (f(x) - P_n(x)) = 0$ for all $x$."

(b) "The series $\sum \frac{n^2}{n!}$ converges."

(c) "If $f'(x) = f(x)$ for all $x$ and $f(0) = 1$ then $f(x) = e^x$ for all $x$."

(d) "If $f$ is three times differentiable on $\mathbb{R}$ and $f(2) \le f(x)$ for all $x$, and if $f''(2) = 0$, then $f'''(2) = 0$."

(2) Compute: (a) $\int \frac{1}{1+e^x} dx$ , (b) $\int_1^{e^2} x^4 \log x dx$ .

(3) (a) Given a non-negative function $f\colon [a, b] \to \mathbb{R}$, write a formula for the volume of the body of revolution that is obtained from the region under the graph $\{0 \le y \le f(x)\}$ by rotating it about the $x$ axis.

(b) The ball of radius $r$ centred at the origin of $\mathbb{R}^3$ can be obtained as the body of revolution of the region under the graph of the function $f(x) = \sqrt{r^2 - x^2}$ by rotating it about the $x$ axis. Calculate its volume using the formula from (a).

(4) Consider the following three facts.

Fact 1: for all $x > -1$, $\log(1 + x) - x = \int_0^x \frac{1}{(1+t)^2}(t - x)dt$.

Fact 2: for all $x \in \mathbb{R}$, $\lim_{n\to\infty} n \left(\log(1 + \frac{x}{n}) - \frac{x}{n}\right) = 0$.

Fact 3: for all $x \in \mathbb{R}$, $\lim_{n\to\infty} (1 + \frac{x}{n})^n = e^x$.

(a) Prove Fact 1. (Hint: integration by parts.)
(b) Assuming Fact 1, prove Fact 2.
(c) Assuming Fact 2, prove Fact 3.

(5) (a) Let $f$ be a function that is $n$ times differentiable at a point $a \in \mathbb{R}$, and let $n \in \mathbb{N}$. Define the Taylor polynomial of order $n$ at $a$ for $f$.

(b) Let $f(x) = (1+x)^\alpha$, for $x > -1$, where $\alpha$ is a real number that is not an integer. Find the Taylor polynomial of order $n$ at 0 for $f$.

(c) Bonus question – **optional**:
Find an $n$ such that the Taylor polynomial of $(1+x)^{1/2}$ of order $n$ at 0, evaluated at $x = 1$, approximates $\sqrt{2}$ with an error of at most $1/100$.