MAT157Y TERM TEST 2 – COMMENTS.

(1) (a) Give one example of a function $f\colon \mathbb{R} \to \mathbb{R}$ that satisfies the following condition:
($\star$) There exists $\delta > 0$ such that for all $\epsilon > 0$, if $0 < |x-1| < \delta$ then $|f(x) - 2| < \epsilon$.
(b) Give one example of a function $f\colon \mathbb{R} \to \mathbb{R}$ that satisfies $\lim_{x\to 1} f(x) = 2$ but that does not satisfy the condition ($\star$).
Do not explain.

Comment on (a): the condition is that f be identically equal to 2 on some punctured neighbourhood of $x = 1$.

(2) (a) Let $A \subset \mathbb{R}$ be a non-empty set of positive numbers. Say whatever you can about $\inf A$ and $\sup A$.
(b) Does the set $\{x \mid x^2 + x - 1 < 0\}$ have a least upper bound? If so, find it. If not, explain in one sentence.

Complete solution to (a):
$\inf A$ exists and is ≥ 0. If $\sup A$ exists, then $\sup A \geq \inf A$ and $\sup A > 0$.

(3) (a) Prove that there is a point x between 0 and 2 such that $x^5 - x = 10$.
(b) Prove that there does not exist a point x between 0 and 2 such that $x^5 - x = -1$. (Hint: find the minimum value of $x^5 - x$ for x between 0 and 2 and show that it is > -1).

Complete solution to (a): The function $f(x) = x^5 - x$ is continuous on the interval $[0, 2]$, and (because $f(0) = 0$ and $f(2) = 30$,) we have $f(0) < 10 < f(2)$. By the intermediate value theorem, there exists x such that $0 < x < 2$ and $f(x) = 10$.

Complete solution to (b): The function $f(x) = x^5 - x$ is continuous on the interval $[0, 2]$, so by the extreme value theorem, it attains a minimum on this interval. Since f is differentiable, if this minimum is attained at a point x in the open interval $(0, 2)$, then at this point we have $f'(x) = 0$.

Since $f'(x) = 5x^4 - 1$ for all x, if $f'(x) = 0$ then $x = \frac{1}{\sqrt[4]{5}}$ or $x = -\frac{1}{\sqrt[4]{5}}$. Of these two points, only the first is in the open interval $(0, 2)$. We calculate the value of f at this point: $(\frac{1}{\sqrt[4]{5}})^5 - \frac{1}{\sqrt[4]{5}} = \frac{1}{5\sqrt[4]{5}} - \frac{1}{\sqrt[4]{5}} = \frac{-4}{5\sqrt[4]{5}}$.

So the minimum value of f on the interval $[0, 2]$ is either $\frac{-4}{5\sqrt[4]{5}}$, or $f(0)$ (which is 0), or $f(2)$ (which is 30). Since $\frac{-4}{5\sqrt[4]{5}}$ is the smallest of these three numbers, the minimum value of f in the interval $[0, 2]$ is $\frac{-4}{5\sqrt[4]{5}}$.

Since $5 > 1$, we have $\sqrt[4]{5} > 1$, and so $\frac{4}{5\cdot\sqrt[4]{5}} < \frac{4}{5}$, and $\frac{-4}{5\sqrt[4]{5}} > -\frac{4}{5}$. So for all $x \in [0, 2]$ we have $x^5 - x \geq \frac{-4}{5\sqrt[4]{5}} > -1$, and in particular $x^5 - x \neq -1$.

(4) (a) Find the equation for the tangent line to the graph of the function $x \mapsto \frac{1}{x}$ at the point $(7, \frac{1}{7})$.
(b) Find the derivative of the function $x \mapsto \sin(\frac{1}{x})$ at the point $x = \frac{6}{\pi}$. Simplify if you can.

Comment on (b): Simplifying the expression requires evaluating $\cos(\frac{\pi}{6})$. If you are not fluent with trigonometry, please review it. I highly recommend the book "Trigonometry" by Gelfand and Saul.

(5) For each of the following statements, determine if it is true or false, and explain informally in one short sentence. Do not give formal proofs.

(a) For any function $f\colon \mathbb{R} \to \mathbb{R}$, if f is differentiable and bounded from below, then there exists $x \in \mathbb{R}$ such that $f(x) \leq f(y)$ for all $y \in \mathbb{R}$.

(b) For any function $f\colon \mathbb{R} \to \mathbb{R}$, if f is differentiable and $f'(x) > 0$ for all x, then there is no x such that $f(x) \leq f(y)$ for all $y \in \mathbb{R}$.

(c) For any function $f\colon \mathbb{R} \to \mathbb{R}$, if f is differentiable and $f'(x) = 0$ for all $x > 0$, then $f(0) = f(100)$.

(d) For any function $f\colon \mathbb{R} \to \mathbb{R}$, if f is differentiable and $f'(x) = 0$ whenever $|x| > 100$, then f has a maximal value.

Solution to (a): False. For example $f(x) = e^x$ is differentiable and bounded from below (by 0) but does not attain a minimum.

Solution to (b): True. At a minimum point x for f we would have had $f'(x) = 0$.

Solution to (c): True. This is a consequence of the mean value theorem.
(*Comment:* We need to apply the mean value theorem only once, to the interval $[0, 100]$, and we don't need to know anything about $f'(0)$.)

Comment on (d): While marking the test I realized that the question can be parsed in two different ways. I allowed both interpretations.

(d') For any function $f\colon \mathbb{R} \to \mathbb{R}$, if

- f is differentiable

and

- $f'(x) = 0$ whenever $|x| > 100$,

then f has a maximal value.

This statement is true. By the extreme value theorem, the restriction of f to the closed interval $[-100, 100]$ has a maximal value; the assumption on the derivative implies that f is constant on each of $[100, \infty)$ and on $(-\infty, -100]$, so the maximal value for f on $[-100, 100]$ is also a maximal value for f on all of $\mathbb{R}$.