MAT157Y TERM TEST 1, OCTOBER 2017

Please answer all the questions. The questions have equal weight. You have 100 minutes. No aids allowed. The list of axioms for the real numbers is attached after the questions.

(1) Let $f\colon \mathbb{R} \to \mathbb{R}$ be any function.
  (a) Define $\lim_{x\to 10} f(x) = 1$.
  (b) State the negation of $\lim_{x\to 10} f(x) = 1$ without using the words "no" or "not".

(2) Prove that the floor function, $f(x) = \lfloor x \rfloor$ ($=$ the largest integer $\leq x$), is not continuous at $x = 2$.

(3) Let $f(x) = x^2$. Let $a = 10$ and $\ell = 100$. Let $\epsilon = 1$. Find a number $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - \ell| < \epsilon$. Show your work, but do not explain.

(4) Assuming the postulates for the real numbers and any relevant definitions, prove that if $a \cdot a = a$ then $a = 0$ or $a = 1$. Justify every step.

(5) For which values of A, B, C, D, E does the equation $Ax^2 + Bx + Cy^2 + Dy + E = 0$ define a circle? Show your work, but do not explain.

(6) Find each of the following limits. Show your work, but do not explain.
  (a) $\lim_{x\to 2} \dfrac{x^3 - 8}{x - 2}$.
  (b) $\lim_{x\to 3} \dfrac{x^3 - 8}{x - 2}$.
  (c) $\lim_{x\to 1} \dfrac{1 - \sqrt{x}}{1 - x}$.

(7) For each of the following functions f, find an even function g and an odd function h such that $f = g + h$. Use as simple expressions as possible for g and h. Show your work, but do not explain.
  (a) $f(x) = (x - 1)^2$.
  (b) $f(x) = e^x$. (You don't need to know what is e except that it's some number between 2 and 3.)
  (c) $f(x) = \sin(x + \pi/6)$.

(8) For each of the following statements, determine if it is true or false, and explain your answer in one sentence.
  (a) If $\lim_{x\to a} f(x) = \ell$ and $\lim_{x\to a} g(x) = m$ and $f(x) < g(x)$ for all x, then $\ell < m$.
  (b) Postulates (P1)–(P12) for the real numbers imply that there exists $x > 0$ such that $x \cdot x = 1 + 1$.
  (c) There does not exist a triangle with sidelengths 2, 3, and 6.

Postulates for the real numbers

(P1) (Associative law for addition) $a + (b + c) = (a + b) + c.$

(P2) (Existence of an additive identity) $a + 0 = 0 + a = a.$

(P3) (Existence of additive inverses) $a + (-a) = (-a) + a = 0.$

(P4) (Commutative law for addition) $a + b = b + a.$

(P5) (Associative law for multiplication) $a \cdot (b \cdot c) = (a \cdot b) \cdot c.$

(P6) (Existence of a multiplicative identity) $a \cdot 1 = 1 \cdot a = a; \quad 1 \neq 0.$

(P7) (Existence of multiplicative inverses) $a \cdot a^{-1} = a^{-1} \cdot a = 1, \text{ for } a \neq 0.$

(P8) (Commutative law for multiplication) $a \cdot b = b \cdot a.$

(P9) (Distributive law) $a \cdot (b + c) = a \cdot b + a \cdot c.$

(P10) (Trichotomy law) For every number a, one and only one of the following holds:

(i) $a = 0$,
(ii) a is in the collection P,
(iii) $-a$ is in the collection P.

(P11) (Closure under addition) If a and b are in P, then $a + b$ is in P.

(P12) (Closure under multiplication) If a and b are in P, then $a \cdot b$ is in P.

A number x is a **least upper bound** of A if

(1) x is an upper bound of A,
and (2) if y is an upper bound of A, then $x \leq y$.

A set A of real numbers is **bounded above** if there is a number x such that

$$x \geq a \quad \text{for every } a \text{ in } A.$$

Such a number x is called an **upper bound** for A.

(P13) (The least upper bound property) If A is a set of real numbers, $A \neq \emptyset$, and A is bounded above, then A has a least upper bound.

from: Spivak, "Calculus"