Quan	tiza	ntion,
after	Sou	ıriau

Souriau Prequantizatio Quantization? Group algebra Classical Quantum Nilpotent Reductive E(3)

Quantization, after Souriau

François Ziegler (Georgia Southern)

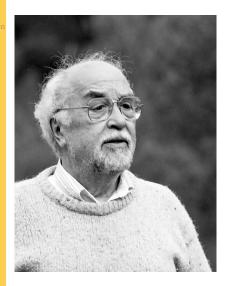
Geometric Quantization: Old and New 2019 CMS Winter Meeting Toronto, 12/8/2019

Abstract: J.-M. Souriau spent the years 1960-2000 in a uniquely dogged inquiry into what exactly quantization is and isn't. I will report on results (of arXiv:1310.7882 etc.) pertaining to the last (still unsatisfactory!) formulation he gave.

J.-M. Souriau

Souriau

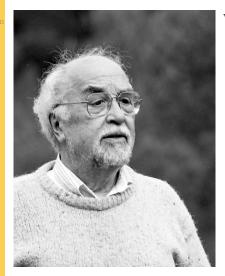
Prequantization Quantization? Group algebra Classical Quantum Nilpotent Reductive E(3)



J.-M. Souriau

Souriau

Prequantization Quantization? Group algebra Classical Quantum Nilpotent Reductive E(3)

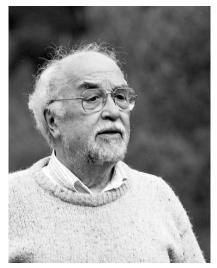


What is quantization?

J.-M. Souriau

Souriau

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What is quantization?

« How do I arrive at the matrix that represents a given quantity in a system of known constitution? »

> — H. Weyl, Quantenmechanik und Gruppentheorie (1927)

Souriau

Prequantization

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Let (X, ω) be a prequantizable symplectic manifold: $[\omega] \in H^2(X, \mathbb{Z})$.

Mantra:

Prequantization constructs a representation of the Poisson algebra $C^{\infty}(X)$, which is "too large" because not irreducible enough.

(We then need "polarization" to cut it down.)

Prequantization

Souriau

Prequantization

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(We then need "polarization" to cut it down.)

Souriau:

Not the point! What prequantization constructs is a group Aut(L) with "Lie algebra" $C^{\infty}(X)$, of which X is a coadjoint orbit.

(Every prequantizable symplectic manifold is a coadjoint orbit, 1985.)

Prequantization

Quantization?

Souriau

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Mantra:

Quantization is some sort of functor from a "classical" category (symplectic manifolds and functions?) to a "quantum" category (Hilbert spaces and self-adjoint operators?).

Besides, it doesn't exist ("by van Hove's no-go theorem").

Quantization?

Souriau

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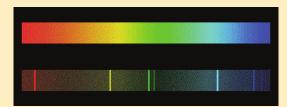
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Souriau:

No! Quantization is a switch from *classical states* to *quantum states*:



Quantization, after Souriau

Souriau Prequantizatio Quantization? Group algebra Classical Quantum Nilpotent Reductive • **C**[G] := {finitely supported functions $G \to C$ } $\ni c = \sum_{g \in G} c_g \delta^g$ is a *-algebra: $\delta^g \cdot \delta^h = \delta^{gh}$, $(\delta^g)^* = \delta^{g^{-1}}$ (and a G-module)

• C[G]' \cong C^G = {all functions $m : G \to C$ }: $\langle m, c \rangle = \sum c_g m(g)$

• G-invariant sesquilinear forms on C[G] write $(c, d) \mapsto \langle m, c^* \cdot d \rangle$ $(\delta^e, g\delta^e) \mapsto m(g)$

Definition, Theorem (GNS, L. Schwartz)
Call *m* a state of G if positive definite: ⟨m, c* · c⟩ ≥ 0, and m(e) = 1.
Then C[G]/C[G]⁻¹ is a unitary G-module, realizable in C[G]' as
GNS_m = {φ ∈ C^G such that ||φ||² := sup_{c∈C[G]} (|(φ,c)|²/(m,c*·c)) < ∞}. *m* is cyclic in GNS_m (its G-orbit has dense span).
Any unitary G-module with a cyclic unit vector φ is GNS_(φ,+φ).

Quantization, after Souriau

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Quantization, after Souriau

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Group algebra

Group algebra. States

Souriau

Example 1: Characters

 $f\,\chi:G\to U(1)$ is a character, then χ is a state and

 $\text{GNS}_{\chi} = \boldsymbol{C}_{\chi}$

 $(= \mathbf{C}$ where G acts by χ).

Example 2: Discrete induction (Blattner 1963)

If n is a state of a subgroup $\mathrm{H}\subset\mathrm{G}$ and m(g)= then m is a state and

(g) if $g \in H$, otherwise

```
\mathrm{GNS}_m = \mathrm{ind}_\mathrm{H}^\mathrm{G} \mathrm{GNS}_n
```

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Prequanti

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Group algebra. States

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Quantization?

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Classical Quantum Nilpotent

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E(3)

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Group algebra. States

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Quantization?

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Classical Quantum Nilpotent

F(2)

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(= **C** where G acts by χ).

Example 2: Discrete induction (Blattner 1963)

If *n* is a state of a subgroup $H \subset G$ and $m(g) = \begin{cases} n(g) & \text{if } g \in H, \\ 0 & \text{otherwise,} \end{cases}$

$$GNS_m = ind_H^G GNS_n$$

Classical (statistical) states

Souriau Prequantization Quantization Group algebra **Classical** Quantum

Reductive E(3) Let X be a coadjoint orbit of G (say a Lie group). Continuous states m of (g, +) correspond to probability measures μ on g^* (Bochner):

$$m(\mathbf{Z}) = \int_{\mathfrak{g}^*} \mathrm{e}^{\mathrm{i}\langle x, \mathbf{Z} \rangle} d\mu(x). \tag{1}$$

Definition

A *statistical state* for X is a state m of g which is concentrated on X, in the sense that its spectral measure (μ above) is.

This works even without assuming continuity of m: in (1), make g discrete and hence replace g^* by its *Bohr compactification*

 $\hat{\mathfrak{g}} = \{ all \text{ characters of } \mathfrak{g} \},\$

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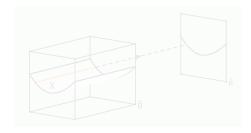
Quantum states

Souriau Prequantization? Quantization? Group algebra Classical Classical Nilpotent Reductive E(3)

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Statistical interpretation: the spectral measure of $m \circ \exp_{|\mathfrak{a}|}$ gives the probability distribution of $x_{|\mathfrak{a}|}$ (or "joint probability" of the Poisson commuting functions $\langle \cdot, Z_j \rangle$ for Z_j in a basis of \mathfrak{a}).



Quantum states

Souriau

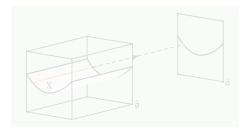
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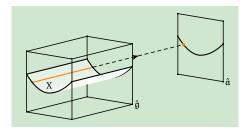
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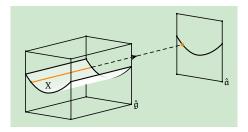
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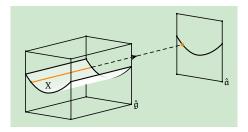
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Souriau Prequantizati Quantization Group algebr Classical Quantum Nilpotent Reductive E(3)

If $V = GNS_m$, then $(\varphi, \cdot \varphi)$ is a quantum state for X for *all* unit $\varphi \in V$. Definition

G-modules V with this property are *quantum representations* for X.

They need not be continuous, nor irreducible on transitive subgroups.

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Suppose a state *n* of a connected Lie group H is quantum for a *point*orbit $\{y\} \subset (\mathfrak{h}^*)^{\mathrm{H}}$. Then *y* is *integral*, and *n* is the character such that $n(\exp(Z)) = e^{i(y,Z)}$ (2)

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Prequantization Quantization? Group algebra Classical **Quantum** Nilpotent Reductive

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Example 2: Prequantization is not quantum

Let L be the prequantization line bundle over $X = (\mathbb{R}^2, dp \land dq)$. The resulting representation of Aut(L) in $L^2(X)$ is not quantum for X.

Sketch of proof:

It represents the flow of the *bounded* hamiltonian $H(p, q) = \sin p$ by a 1-parameter group whose self-adjoint generator is *unbounded* — it's equivalent to multiplication by $\sin p + (k - p) \cos p$ in $L^2(\mathbb{R}^2, dp \, dk)$.

Remark

We are rejecting this representation for *spectral* reasons. Unlike van Hove who rejected it for being *reducible* on the Heisenberg subgroup, we can still hope that Aut(L) has a representation quantizing X.

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Quantum states

Souriau

Prequantization Quantization? Group algebra Classical Quantum

Reductive E(3)

On the other hand...

Theorem (Howe-Z., Ergodic Theory Dynam. Systems 2015)

- G noncompact simple: every nonzero coadjoint orbit has $bX = bg^*$.
- G connected nilpotent: every coadjoint orbit has the same Bohr closure as its affine hull.

- G noncompact simpler every unitary representation is quantum for every nonzero condipint orbit.
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Eigenstates in nilpotent groups

Souriau

- Prequantization
- Quantization
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- Classical
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Eigenstates in nilpotent groups

Quantization, after Souriau

Souriau Prequantization Quantization? Group algebra Classical Quantum Nilpotent

G : connected, simply connected nilpotent Lie group,

- X : coadjoint orbit of G,
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A connected subgroup $\mathrm{H}\subset\mathrm{G}$ is subordinate to x if, equivalently,

Eigenstates in nilpotent groups

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Nilpotent Reductive

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 - $\langle x, [\mathfrak{h}, \mathfrak{h}] \rangle = 0$
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- Prequantizatio Quantizatio Group algel Classical Quantum Nilpotent
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Theorem

Let $H \subset G$ be maximal subordinate to $x \in X$. Then there is a unique quantum eigenstate for X belonging to $\{x_{|\mathfrak{h}}\} \subset \mathfrak{h}^*$, namely

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Moreover $GNS_m = ind(x, H) := ind_H^G e^{ix \circ \log}_{|H|}$ (discrete induction)

11/18

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Nilpotent

Eigenstates in nilpotent groups

- G : connected, simply connected nilpotent Lie group,
- X : coadjoint orbit of G,
- x : chosen point in X.
- A connected subgroup $H \subset G$ is *subordinate to* x if, equivalently,
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Kirillov (1962) used $\operatorname{Ind}(x, H) := \operatorname{Ind}_{H}^{G} e^{ix \circ \log}_{|H|}$ (usual induction). This is

irreducible \Leftrightarrow H is a *polarization at* x (: subordinate subgroup such that the bound dim(G/H) $\ge \frac{1}{2}$ dim(X) is attained);

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Prequantization Quantization? Group algebra Classical Quantum Nilpotent Reductive

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Eigenstates in reductive groups

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- Prequantization
- Quantization
- Group algebra
- Classical
- Quantum
- Nilpotent

Reductive

E(3)

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G : linear reductive Lie group (: \subset **GL**_n(**R**), stable under transpose)

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Remark: The coadjoint orbit $G(x) = Ind_0^G \{x_{ig}\}$ (symplectic induction).

Conjecture

There is a unique state m of G that extends χ , namely

 $oldsymbol{n}(g) = \left\{egin{array}{cc} \chi(g) & ext{if } g \in \mathrm{Q}, \ \end{array}
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Theorem: The conjecture is true for $G = SL_2(R)$ or $SL_3(R)$, Q Borel.

Souriau

Prequantization? Quantization? Group algebra Classical Quantum Nilpotent Reductive E(3)

Euclid's group G = $\left\{g = \begin{pmatrix} A & c \\ 0 & 1 \end{pmatrix} : A \in SO(3) \atop c \in \mathbb{R}^3 \right\}$

Example: TS²

G acts naturally and symplectically on the manifold $X \simeq TS^2$ of oriented lines (a.k.a. light rays) in \mathbb{R}^3 . 2 formula

 $\omega = k \; d \langle oldsymbol{u}, oldsymbol{dr}
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The moment map

 $\Phi(u,r) = \binom{r \times ku + su}{ku}$

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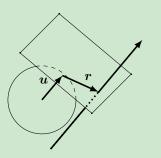
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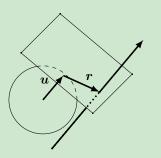
G acts naturally and symplectically on the manifold $X \simeq TS^2$ of oriented lines (a.k.a. light rays) in \mathbb{R}^3 . 2-form_{k.s}:

 $\omega = k \ d\langle u, dr
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The moment map

$$\Phi(u,r) = egin{pmatrix} r imes ku + su \ ku \end{pmatrix}$$

makes X into a coadjoint orbit of G.



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Prequantizatio Quantization? Group algebra Classical Quantum Nilpotent Reductive E(3)

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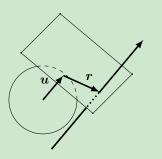
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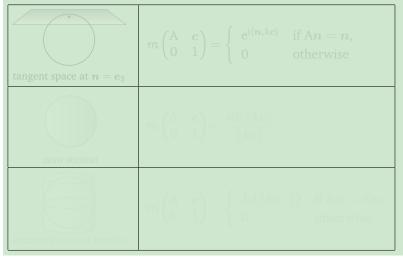
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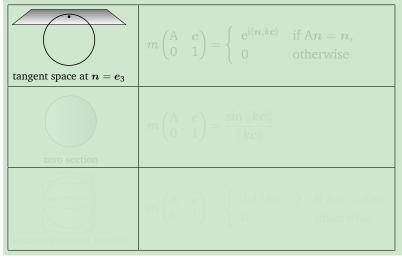


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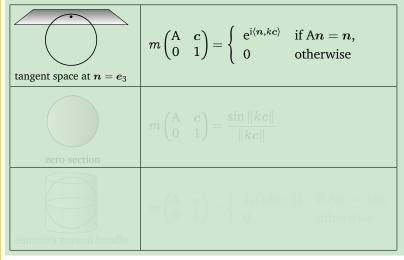


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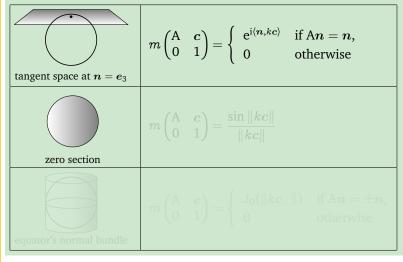


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Image: tangent space at
$$n = e_3$$
 $m \begin{pmatrix} A & c \\ 0 & 1 \end{pmatrix} = \begin{cases} e^{i\langle n, kc \rangle} & \text{if } An = n, \\ 0 & \text{otherwise} \end{cases}$ Image: tangent space at $n = e_3$ $m \begin{pmatrix} A & c \\ 0 & 1 \end{pmatrix} = \frac{\sin ||kc||}{||kc||}$ Image: tangent space at $n = e_3$ $m \begin{pmatrix} A & c \\ 0 & 1 \end{pmatrix} = \frac{\sin ||kc||}{||kc||}$ Image: tangent space at $n = e_3$ $m \begin{pmatrix} A & c \\ 0 & 1 \end{pmatrix} = \begin{cases} J_0(||kc_1||) & \text{if } An = \pm n, \\ 0 & \text{otherwise} \end{cases}$ Image: tangent space at $n = e_3$ $m \begin{pmatrix} A & c \\ 0 & 1 \end{pmatrix} = \begin{cases} J_0(||kc_1||) & \text{if } An = \pm n, \\ 0 & \text{otherwise} \end{cases}$

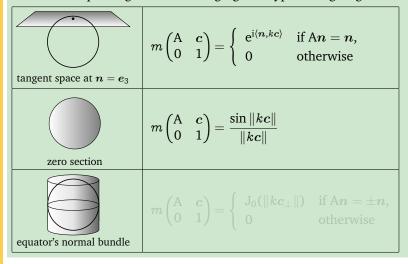
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We have unique* eigenstates belonging to 3 types of lagrangians:



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Prequantizati Quantization Group algebr Classical Quantum Nilpotent Reductive E(3)

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Case s = 0:

The resulting GNS modules can be realized as solution spaces of the Helmholtz equation

$$\Delta \psi + k^2 \psi = 0 \tag{3}$$

with scalar field G-action $(g\psi)(r) = \psi(A^{-1}(r-c))$ and cyclic vectors:



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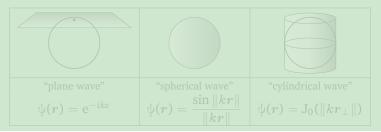
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E(3)

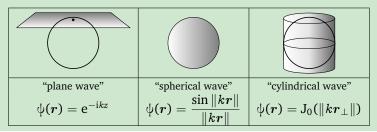
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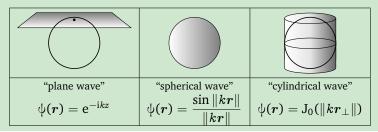
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Case s = 1 (zero section is no longer lagrangian):

The unique eigenstate belonging to the tangent space at $m{n}$ becomes

$$m\begin{pmatrix} A & c \\ 0 & 1 \end{pmatrix} = \begin{cases} e^{i\alpha}e^{i\langle n,kc \rangle} & \text{if } A = e^{j(\alpha n)}, \\ 0 & \text{otherwise} \end{cases} \quad (j(\alpha) := \alpha \times \cdot).$$

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$${
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m S}^2} {
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Souriau

- Prequantization
- Quantization
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- Classical
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- Reductiv
- E(3)

End!