Introduction to the SK Model

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April 2, 2021

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Introduction

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Divide students into two dorms by assigning labels $\sigma_i \in \{-1, +1\}$:

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Dean's problem: maximize the comfort function

$$C(\sigma) = \sum_{i,j=1}^{N} g_{ij}\sigma_i\sigma_j = \sum_{i \sim j} g_{ij} - \sum_{i \not\sim j} g_{ij}$$

over configurations $\sigma \in \Sigma_N$.

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The Hamiltonian of the Sherrington-Kirkpatrick model is the Gaussian process

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Question: What can we say about the maximum $\max_{\sigma\in\Sigma_N} H_N(\sigma)$ asymptotically for systems of large size $N \to \infty$?

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where $R_{1,2} = \frac{1}{N}\sum_{i=1}^{N}\sigma_i^1\sigma_i^2$ is the overlap between σ^1 and σ^2.

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$$\lim_{N\to\infty}\left|\frac{1}{N}\max_{\sigma\in\mathbf{\Sigma}_N} H_N(\sigma) - \frac{1}{N}\mathbb{E}\max_{\sigma\in\mathbf{\Sigma}_N} H_N(\sigma)\right| = 0.$$

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Question: What can we say about $\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\max_{\sigma\in\Sigma_N} H_N(\sigma)$.

An Idea From Physics

Instead of studying $\max_{\sigma \in \Sigma} H_N(\sigma)$ first try to compute the limit of its 'smooth approximation'

$$\lim_{N \to \infty} \frac{1}{N\beta} \mathbb{E} \log \sum_{\sigma \in \Sigma_N} \exp \beta H_N(\sigma)$$

for every inverse temperature parameter $\beta > 0$.

The Free Energy

Definition

The partition function and the free energy are

$$Z_N(\beta) = \sum_{\sigma \in \Sigma_N} \exp \beta H_N(\sigma) \text{ and } F_N(\beta) = \frac{1}{N} \mathbb{E} \log Z_N(\beta).$$

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Theorem (From free energy to maximum)

If the limit $F(\beta) = \lim_{N\to\infty} F_N(\beta)$ exists for every $\beta > 0$, then

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E} \max_{\sigma \in \Sigma_N} H_N(\sigma) = \lim_{\beta\to\infty} \frac{F(\beta)}{\beta}.$$

Proof

- Clearly,

$$\frac{1}{N}\mathbb{E}\max_{\sigma\in\boldsymbol{\Sigma}_N} H_N(\sigma) \leq \frac{1}{N\beta}\mathbb{E}\log\sum_{\sigma\in\boldsymbol{\Sigma}_N}\exp\beta H_N(\sigma) \leq \frac{\log 2}{\beta} + \frac{1}{N}\mathbb{E}\max_{\sigma\in\boldsymbol{\Sigma}_N} H_N(\sigma).$$

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- This means

$$\Big|\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\max_{\sigma\in\Sigma_N} H_N(\sigma) - \frac{F(\beta)}{\beta}\Big| \leq \frac{\log 2}{\beta}.$$

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- By Hölder's inequality for sums $\beta \mapsto \beta^{-1}(F_N(\beta) - \log(2))$ is increasing:

$$\lim_{\beta\to\infty}\beta^{-1}F(\beta)$$

exists. ■

Conclusion

To understand the Dean's problem study the limit of the free energy

$$\lim_{N\to\infty} F_N(\beta)$$

in the SK model.

The Gibbs Measure

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The Gibbs measure is the random probability measure on Σ_N

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Average with respect to $G_N^{\otimes\infty}$ is denoted $\langle\cdot\rangle$.

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$$\frac{\mathrm{d}^2}{\mathrm{d}\beta^2}F_N(\beta) = \frac{1}{N}\Big(\mathbb{E}\langle H_N(\sigma)^2\rangle - \mathbb{E}\langle H_N(\sigma)\rangle^2\Big).$$

Computing Gibbs Averages

- Consider jointly Gaussian vectors $(x(\sigma))$ and $(y(\sigma))$ indexed by countably infinite set Σ with $\mathbb{E}x(\sigma)^2, \mathbb{E}y(\sigma)^2 \leq a$.

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Theorem (Gaussian integration by parts for Gibbs averages)

If $C(\sigma^1, \sigma^2) = \mathbb{E}x(\sigma^1)y(\sigma^2)$, then

$$\mathbb{E}\langle x(\sigma)\rangle = \mathbb{E}\langle C(\sigma^1, \sigma^1) - C(\sigma^1, \sigma^2)\rangle.$$

Proof

- By approximation take Σ finite: $\mathbb{E}\langle x(\sigma)\rangle = \sum_{\sigma^1\in\Sigma} \mathbb{E}x(\sigma^1)G'(\sigma^1)$.

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Lemma (Gaussian integration by parts)

If g is a Gaussian vector in $\mathbb{R}^n$ with $\mathbb{E}|\nabla F(g)| < \infty$, then

$$\mathbb{E}g_1F(g) = \sum_{1\le l\le n}\mathbb{E}g_1g_l\mathbb{E}\partial_{x_l}F(g).$$

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- Take $g = (x(\sigma^1), y(\sigma^1), (y(\sigma^2))_{\sigma^2\in\Sigma})$ and $F(g) = \frac{\exp(y(\sigma^1))}{Z}G(\sigma^1)$:

$$\frac{\partial F}{\partial y(\sigma^1)} = G'(\sigma^1), \qquad \frac{\partial F}{\partial y(\sigma^2)} = -G'(\sigma^1)G'(\sigma^2).$$

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■

Existence of Limiting Free Energy

Guerra-Toninelli

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Lemma (Fekete lemma)

If $(x_n)_{n=1}^{\infty}$ *is a superadditive sequence* $(x_n + x_m \leq x_{n+m})$ *then* $\lim_{n\to\infty} n^{-1} x_n$ *exists.*

Proof

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- For $t \in [0, 1]$ consider the Hamiltonian on Σ_{N+M} defined by

$$H_t(\sigma) = \sqrt{t}H_{N+M}(\sigma) + \sqrt{1-t}(H_N(\rho) + H_M(\tau)).$$

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- $\varphi(0) = \frac{N}{N+M} F_N + \frac{M}{N+M} F_M$ and $\varphi(1) = F_{N+M}$.
- Want to prove $\varphi(0) \leq \varphi(1)$ or $\varphi'(t) \geq 0$.

Proof

- Gaussian integration by parts:

$$\varphi'(t) = \frac{1}{N+M}\mathbb{E}\left\langle \frac{\partial H_t}{\partial t}(\sigma) \right\rangle_t = \frac{1}{N+M}\mathbb{E}\left\langle C(\sigma^1,\sigma^1) - C(\sigma^1,\sigma^2) \right\rangle_t,$$

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- If $R_{1,2} = \frac{\sigma^1\cdot\sigma^2}{N+M}$, $R^-_{1,2} = \frac{\rho^1\cdot\rho^2}{N}$ and $R^+_{1,2} = \frac{\tau^1\cdot\tau^2}{M}$:

$$C(\sigma^1,\sigma^2) = \frac{1}{2}\Big(\mathbb{E}H_{N+M}(\sigma^1)H_{N+M}(\sigma^2) - \mathbb{E}H_N(\rho^1)H_N(\rho^2) - \mathbb{E}H_M(\tau^1)H_M(\tau^2)\Big)$$

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- Convexity of $x \mapsto x^2$:

$$\varphi'(t) = -\frac{1}{2}\mathbb{E}\left\langle R_{1,2}^2 - \frac{N}{N+M}(R^-_{1,2})^2 - \frac{M}{N+M}(R^+_{1,2})^2 \right\rangle_t \geq 0.$$

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Recap

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- Showed $\lim_{N\to\infty} F_N(\beta)$ exists for every $\beta > 0$.

Recap

- Introduced the SK model using the Dean's problem.
- Showed the Dean's problem can be understood by computing $\lim_{N\to\infty} F_N(\beta)$ for every $\beta > 0$.
- Showed $\lim_{N\to\infty} F_N(\beta)$ exists for every $\beta > 0$.
- **Problem:** The Dean still has no idea how to assign dorms...

The Parisi Formula

Historical Overview

- Formula for $\lim_{N\to\infty} F_N(\beta)$ proposed by Sherrington and Kirkpatrick in their original paper.

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Historical Overview

- Formula for $\lim_{N\to\infty} F_N(\beta)$ proposed by Sherrington and Kirkpatrick in their original paper.
- Their solution exhibited 'unphysical behavior' at low temperature.
- Correct formula for $F(\beta)$ at all temperatures was famously discovered by Parisi in 1979.
- First rigorous proof given in 2006 by Talagrand.

Historical Overview

- Formula for $\lim_{N\to\infty} F_N(\beta)$ proposed by Sherrington and Kirkpatrick in their original paper.
- Their solution exhibited ‘unphysical behavior’ at low temperature.
- Correct formula for $F(\beta)$ at all temperatures was famously discovered by Parisi in 1979.
- First rigorous proof given in 2006 by Talagrand.
- More robust proof given in 2014 by Panchenko following his famous proof of the Parisi ultrametricity conjecture.

Parisi Functional

- Given $r \geq 1$ consider two sequences of parameters

$$0 = \zeta_{-1} < \zeta_0 < \zeta_1 < \ldots < \zeta_{r-1} < \zeta_r = 1$$
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- Given i.i.d. standard Gaussian random variables $(\eta_p)_{1 \leq p \leq r}$ let

$$X_r^\zeta = \log 2 \cosh \left(\sum_{1 \leq p \leq r} \sqrt{2}\beta (q_p - q_{p-1})^{1/2} \eta_p \right)$$

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Definition

The Parisi functional is

$$\mathcal{P}(\zeta) = X_0^\zeta - \frac{\beta^2}{2} \sum_{0 \leq p \leq r-1} (q_{p+1}^2 - q_p^2)\zeta_p = X_0^\zeta - \beta^2 \int_0^1 t\zeta(t)\mathrm{d}t.$$

Parisi Formula

Theorem (The Parisi formula)

The limit of the free energy in the SK model is given by

$$\lim_{N\to\infty} F_N = \inf_{\zeta} \mathcal{P}(\zeta),$$

where the infimum is taken over all discrete functional order parameters $\zeta \in \mathcal{D}[0,1]$.

The Ruelle Probability Cascades

The Object

- Random measure on separable Hilbert space.

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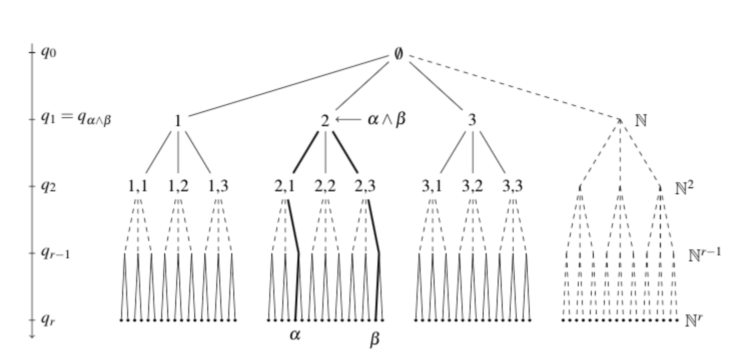

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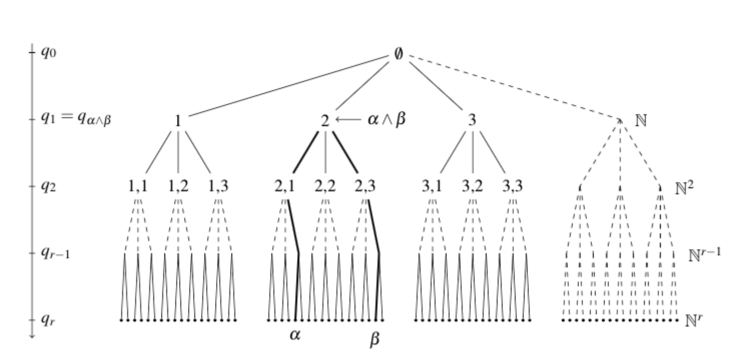

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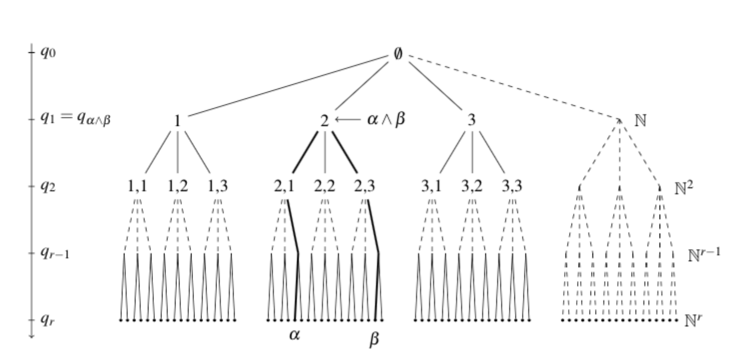

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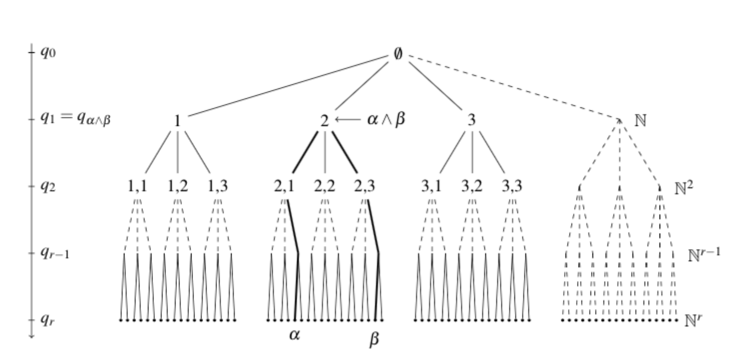

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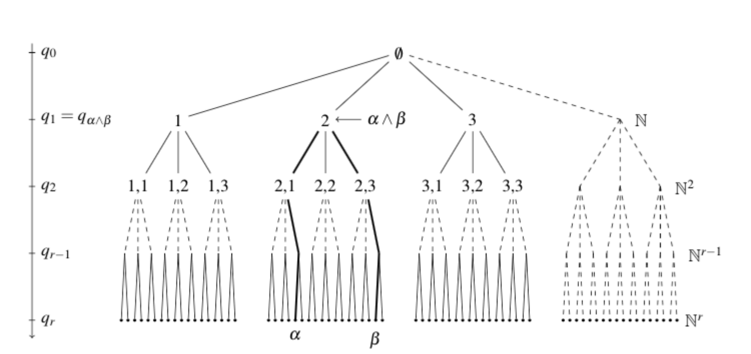

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- $\alpha \wedge \beta = |p(\alpha) \cap p(\beta)|$ and $|\alpha| = |p(\alpha)|$.

The Support

Definition

If (e_α) is an orthonormal sequence in H indexed by $\alpha \in \mathcal{A} \setminus \mathbb{N}^0$, the support of $\mathrm{RPC}(\zeta)$ is the set (h_α) defined by

$$h_\alpha = \sum_{\beta \in p(\alpha)} e_\beta (q_{|\beta|} - q_{|\beta|-1})^{1/2}.$$

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Lemma (RPC overlaps)

For all $\alpha, \beta \in \mathbb{N}^r$ we have $h_\alpha \cdot h_\beta = q_{\alpha \wedge \beta}$.

The Weights

- For $\alpha \in \mathcal{A} \setminus \mathbb{N}^r$, let Π_α be a Poisson process on $(0, \infty)$ with mean measure

$$\mu_{|\alpha|}(\mathrm{d}x) = \zeta_{|\alpha|} x^{-1-\zeta_{|\alpha|}} \mathrm{d}x.$$

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- For $\alpha \in \mathcal{A} \setminus \mathbb{N}^0$ define $w_\alpha = \prod_{\beta \in p(\alpha)} u_\beta$. The weights of $\mathrm{RPC}(\zeta)$ are
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Theorem (RPC averages)

For $\zeta \in \mathcal{D}[0,1]$

$$X_0 = \mathbb{E}\log\langle \exp X_r((\omega_\beta)_{\beta\in p(\alpha)})\rangle = \mathbb{E}\log \sum_{\alpha\in\mathbb{N}^r} \nu_\alpha \exp X_r((\omega_\beta)_{\beta\in p(\alpha)}),$$

where $\langle\cdot\rangle$ *denotes the average with respect to* $\mathrm{RPC}(\zeta)$.

RPC and Parisi Formula

- Consider Gaussian processes $(Z(h_\alpha))_{\alpha \in \mathbb{N}^r}$ and $(Y(h_\alpha))_{\alpha \in \mathbb{N}^r}$ with

$$\mathbb{E}Z(h_{\alpha^1})Z(h_{\alpha^2}) = 2h_{\alpha^1} \cdot h_{\alpha^2} = 2q_{\alpha^1 \wedge \alpha^2}$$
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Theorem (Parisi functional in terms of RPC)

For $\zeta\in\mathcal{D}[0,1]$

$$\mathcal{P}(\zeta) = \mathbb{E}\log\sum_{\alpha\in\mathbb{N}^r}\nu_\alpha 2\cosh\beta Z(h_\alpha) - \mathbb{E}\log\sum_{\alpha\in\mathbb{N}^r}\nu_\alpha\exp\beta Y(h_\alpha)$$
$$= \mathbb{E}\log\langle 2\cosh\beta Z(\sigma)\rangle - \mathbb{E}\log\langle\exp\beta Y(\sigma)\rangle,$$

where $\langle\cdot\rangle$ denotes the average with respect to $\mathrm{RPC}(\zeta)$.

Parisi Formula Upper Bound

Guerra RSB Interpolation

Theorem (Guerra's RSB bound)

For any $\zeta \in \mathcal{D}[0,1]$ *and every* $N \in \mathbb{N}$ *we have* $F_N \leq \mathcal{P}(\zeta)$.

Proof

- For $1 \leq i \leq N$ let $(Z_i(h_\alpha))$ and $(Y_i(h_\alpha))$ be independent copies of $(Z(h_\alpha))$ and $(Y(h_\alpha))$.

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$$H_{N,t}(\sigma,\alpha) = \sqrt{t}H_N(\sigma) + \sqrt{1-t}\sum_{i=1}^{N} Z_i(h_\alpha)\sigma_i + \sqrt{t}\sum_{i=1}^{N} Y_i(h_\alpha).$$

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$$\varphi(t) = \frac{1}{N} \mathbb{E} \log \sum_{\sigma,\alpha} \nu_\alpha \exp \beta H_{N,t}(\sigma,\alpha).$$

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- Want to prove $\varphi(1) \leq \varphi(0)$ or $\varphi'(t) \leq 0$.

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- Gaussian integration by parts:

$$\varphi'(t) = \frac{1}{N}\mathbb{E}\left\langle \frac{\partial H_{N,t}(\rho)}{\partial t} \right\rangle_t = \frac{1}{N}\mathbb{E}\langle C(\rho^1, \rho^1) - C(\rho^1, \rho^2)\rangle_t,$$

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- Since $C(\rho^1,\rho^1) = 0$ this shows $\varphi'(t) \le 0$. ■

The End

Thank you!

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