A Mass-Conserving Toy Model of Blood Pulses in Arterial Networks

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Presentation Overview

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1. Statement of model PDE and physical motivation
2. Adapting the model PDE to a network
3. Local well-posedness of network model
4. Energy method for global well-posedness of network model
5. Some numerical experiments

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Model setup

We study blood flow in a viscoelastic artery that is perfectly cylindrical in equilibrium

- x = axial displacement along artery (artery is very long so $x \in \mathbb{R}$)
- t = time
- $u(x,t)$ = deviation from equilibrium of artery's cross-sectional area
- Real parameters: $\mu, \alpha, \nu, \geq 0,\ \gamma \in [0,1]$, and $p \in \mathbb{N}$

We suppose u solves the PDE

$$\left(1-\mu^2\partial_{xx}\right)u_t + \partial_x\left(\alpha u + \frac{\gamma}{p+1}u^{p+1}\right) - \nu u_{xx} = 0$$

called the **generalized Benjamin-Bona-Mahony-Burgers equation (gBBMB)**

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Why gBBMB?

- Benjamin et al. 1972: original Benjamin-Bona-Mahony eqn. (BBM) analyzed, stressed as a more realistic substitute for Korteweg-de Vries eqn. (KdV)
- BBM is second-order in space (vs. third order KdV) and features short linear waves w/ bounded group velocity
- Erbay et al. 1992: generalized KdV-Burgers eqn. models blood vessel motion even when the artery walls are nonlinearly elastic
- I have taken Erbay et al.'s model and replaced KdV dispersion with BBM dispersion in light of the first two points
- Nonlinear dispersive models should work best in very long arteries, particularly the femoral artery

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Flow in a Network of Arteries

Question: How do bifurcations of arteries or changes in artery elasticity (due to arteriosclerosis or stents) affect pulsatile blood flow?

- To address question, we formulate our model on a **network**, a set of finitely many intervals glued together at various junctions
- Coefficients of model PDE are allowed to vary from edge to edge to reflect changes in artery elasticity
- Need to specify compatibility conditions at junctions

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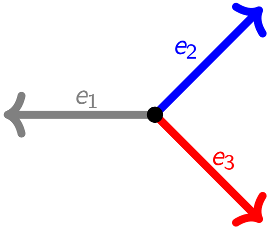


Stents in the Femoral Artery

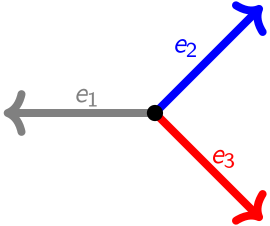
- As recently as 2017, use of stents in upper portion of femoral artery remains contentious (see for instance TASC II 2007 vascular surgery advisory vs. Goueffic et al. 2017)
- Mathematical modelling may be useful to determine best practices
- gBBMB is much too simple to give definitive answers, but may be a good toy model for basic wave-stent (or wave-sclerosis) interactions
- Predictions made w/ gBBMB should be be benchmarked against simulations of the primitive equations to discover when gBBMB is a good enough substitute over long length scales (not addressed by me)

Comparison with Related Work

- BBM has been studied on networks previously, most notably by Bona and Cascaval (B & C) in 2008
- B & C formulation does not guarantee conservation of total mass of blood in general, though my formulation takes care of this
- My proof of local well-posedness is a fixed-point argument identical to B & C, but I provide an explicit proof of global well-posedness by energy methods as well. I believe the explicit energy equation is really worth looking at.

Model Setup 1

- Network X consists of N edges e_i
- One incoming edge $e_1 = (-\infty, 0]$, from which a signal arrives at a central **junction** (the point $x = 0$) and scatters off into the other edges, which are all copies of $[0, \infty)$



Model Setup 2

- Solution to gBBMB on network X: an array of N functions $u_i(x, t)$, with u_i satisfying gBBMB on the edge e_i

- **Issue**: each $u_i(x, t)$ needs to be assigned a boundary value at $x = 0$ to have a hope of well-posedness... thus since we have $i = 1, ..., N$ we need to impose N conditions at the junction.

Continuity Condition

For all $i, j = 1, ..., N$ and all t, we have $u_i(0, t) = u_j(0, t)$.

- Continuity only provides $N - 1$ independent conditions. Need one more!

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Model Setup 3

- Coeffs. of PDE now vary with i, so we write them as $\mu_i, \alpha_i, \nu_i \geq 0,\ \gamma_i \in [0,1]$
- Denote **advective flux** on the edge e_i by

$$f_i(u_i) = \alpha_i u_i + \frac{\gamma_i}{p+1} u_i^{p+1}$$

- Define the **total flux** on the edge e_i by

$$F_i(u_i) = -\mu_i^2 u_{i,xt} + f_i(u_i) - \nu_i u_{i,x}$$

- gBBMB on e_i can now be written in conservative form as

$$u_{i,t} + \partial_x \left(F_i(u_i)\right) = 0$$

Model Setup 4

- Physically, $M = \sum_i \int_{e_i} u_i$ represents the total "normalized" volume of blood in the arterial network X... can also take this as total mass of blood assuming blood has unit mass density

- As anyone who has squished a water balloon knows, $\frac{\mathrm{d}M}{\mathrm{d}t} = 0$ (mass must be conserved)!

- A quick check with the conservative form of gBBMB shows that mass is conserved provided we impose the following final junction condition:

Mass Conservation Condition

$$F_{\text{in}}(u_{\text{in}})\big|_{x=0} = \sum \left[F_{\text{out}}(u_{\text{out}})\right]_{x=0},$$

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Model Setup: Summary

X a network with edges e_i ($i = 1, ..., N$), then we seek functions $u_i(x,t)$ defined for $(x,t) \in e_i \times [0,\infty)$ (with suitable regularity) satisfying the system

$$0 = \left(1 - \mu_i^2 \partial_x^2\right) u_{i,t} + \partial_x \left(\alpha_i u_i + \frac{\gamma_i}{p+1} u_i^{p+1} \right) - \nu_i u_{i,xx}$$

$$+ \;\; u_i\big|_{x=0} = u_j\big|_{x=0} \;\forall\; i,j \text{ (continuity at junction)}$$

$$+ \;\; F_{\text{in}}(u_{\text{in}})\Big|_{x=0} = \sum \left[F_{\text{out}}(u_{\text{out}})\right]_{x=0} \text{ (mass conservation at junction)}$$

$+$ initial conditions.

Local Well-Posedness Preamble

- Now that we have a physically sensible model, must establish local-in-time well-posedness (LWP) of this model (existence and uniqueness of solutions, cts. dependence on initial data).
- Start by reviewing how to write gBBMB on a half-line $[0, \infty)$ as a fixed-point problem
- Then, use the above together w/ junction conditions to write gBBMB on a network as a fixed-point problem (requires us to define function spaces on our network)

Some Function Spaces

- $C_b^k(U)$ = real-valued functions on $U \subseteq \mathbb{R}^n$ whose derivatives up to order k are cts. and bounded; this becomes a Banach space when endowed with sup-norm

- Given any Banach space A, $C_b(0,T;A)$ denotes the Banach space of all continuous functions $u\colon [0,T] \to A$ equipped with the norm
$$\|u\|_{C_b(0,T;A)} = \sup_{[0,T]} \|u(t)\|_A. \quad (0.1)$$

- Also need less common fnc. spaces:

Half-Line gBBMB Phase Space

$$\mathcal{B}_T^{k,\ell} \doteq \left\{ u \mid \forall\ i \in [0,k],\ j \in [0,\ell],\ \partial_t^k \partial_x^\ell u \in C_b\left([0,\infty) \times [0,T]\right) \right\}$$

gBBMB on a Half-Line

- Given $h(t) \in C_b[0,\infty)$, $\varphi(x) \in C_b[0,\infty)$, find $T > 0$ and $u(x,t) \in \mathcal{B}_T^{1,2}$ such that

$$\begin{aligned} (1-\mu^2\partial_x^2)u_t + (f(u))_x - \nu u_{xx} = 0 \quad & \forall\ (x,t) \in (0,\infty)\times(0,T), \\ u(0,t) = h(t) \quad & \forall\ t \in [0,T), \\ u(x,0) = \varphi(x) \quad & \forall\ x \in [0,T). \end{aligned}$$

- At least formally,

$$u_t = (1-\mu^2\partial_x^2)^{-1}\left[-(f(u))_x + \nu u_{xx}\right].$$

  Treat the above expression as an ODE for u

- Once we find an explicit expression for $(1-\mu^2\partial_x^2)^{-1}$, solve the ODE and determine a nonlinear operator for which u arises as a fixed point.

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Inverting $1 - \mu^2\partial_x^2$

Lemma (Green's Function for $1 - \mu^2\partial_x^2$)

Let $\delta(x-y)$ denote the Dirac distribution centred at $y \in \mathbb{R}$. The function

$$G(x,y) \doteq -\frac{1}{2\mu}\left(e^{\frac{-(x+y)}{\mu}} - e^{-\frac{|x-y|}{\mu}}\right) : [0,\infty)^2 \to \mathbb{R}$$

satisfies the PDE

$$\left(1-\mu^2\partial_x^2\right) G(x,y) = \delta(x-y) \quad \forall\ (x,y) \in (0,\infty)^2$$

in the sense of distributions, with $G(0,y) = 0$ and $\lim_{x\to\infty} G(x,y) = 0\ \forall\ y \in [0,\infty)$.

Thus we have

$$(1-\mu^2\partial_x^2)^{-1} g(x) = \int_0^\infty G(x,y)\, g(y)\ \mathrm{d}y.$$

Fixed-Point form of gBBMB on Half-Line

- Define some auxiliary objects by

$$K(x,y) \doteq \frac{1}{2\mu^2}\left(e^{\frac{-(x+y)}{\mu}} + \mathrm{sgn}(x-y)\ e^{-\frac{|x-y|}{\mu}}\right),$$

$$\mathbb{B}_{\mathrm{adv}}[u](x,t) \doteq \int_0^t \int_0^\infty e^{-\frac{\nu}{\mu^2}(t-s)}\ K(x,y)\ f(u(y,s))\,\mathrm{d}y\ \mathrm{d}s,$$

$$\mathbb{B}_{\mathrm{visc}}[u](x,t) \doteq \frac{\nu}{\mu^2}\int_0^t \int_0^\infty e^{-\frac{\nu}{\mu^2}(t-s)}\ G(x,y)\ u(y,s)\,\mathrm{d}y\ \mathrm{d}s,$$

Note that $G(x,y), K(x,y)$ are rapidly decaying! Important later.

- Solve linear ODE, integrate by parts $\Rightarrow$ fixed-pt. formulation of gBBMB now becomes

$$u(x,t) = e^{-\frac{\nu t}{\mu^2}}\varphi(x) + \left(h(t) - h(0)e^{-\frac{\nu t}{\mu^2}}\right) e^{-\frac{x}{\mu}} + \mathbb{B}_{\mathrm{adv}}[u](x,t) + \mathbb{B}_{\mathrm{visc}}[u](x,t).$$

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Function Spaces on a Network

Definition

$$C_b(X) \doteq \{(u_1, \ldots, u_N) \in (C_b[0,\infty))^N \mid u_1(0) = \cdots = u_N(0)\},$$
$$H^1(X) \doteq \left(H^1(0,\infty)\right)^N \cap C_b(X).$$

$C_b(X)$ becomes a Banach space when equipped with the norm

$$\|u\|_{C_b(X)} \doteq \max_{i=1,2,3} \|u_i\|_{C_b[0,\infty)}.$$

Additionally, $H^1(X)$ becomes a Hilbert space when equipped with the sum inner product.

Cauchy Problem for gBBMB on a Network

- For some $T > 0$ and a given $\varphi \in C_b(X)$, find

$$u \in C_b([0, T]; C_b(X))$$

such that if $u_i \doteq u|_{e_i}$ then

$$\begin{aligned} \left(1 - \mu_i^2 \partial_x^2\right) u_{i,t} + (f_i(u_i))_x - \nu_i u_{i,xx} &= 0, \;\; i = 1, \dots, N, \\ u_i(0,t) &= u_j(0,t), \\ F_{\text{in}}(u_{\text{in}})\big|_{x=0} - \sum [F_{\text{out}}(u_{\text{out}})]_{x=0} &= 0, \\ u_i(x,0) &= \varphi_i(x). \end{aligned}$$

Proposition (LWP for Network Problem)

Given $\varphi \in C_b(X)$ with $\varphi_i \in C_b^2[0, \infty)$ for each i, there exists $T > 0$ and a unique u satisfying the above (classically). That is, $u_i \in \mathcal{B}_T^{1,2}$ for all i. Additionally, u depends continuously on the initial data φ.

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Proof of LWP

- Using the same Green's function strategy applied to the half-line problem, we can write gBBMB on a network as a fixed point problem.

- After some calculation leveraging *both* junction conditions (see lecture notes),

$$u_i(x,t) = e^{-\frac{\nu_i t}{\mu_i^2}} \varphi_i(x) + \left(\Phi[u] + \varphi(0) \left(1 - e^{-\frac{\nu_i t}{\mu_i^2}} \right) \right) e^{-\frac{x}{\mu_i}}$$
$$+ \sigma_i \mathbb{B}_{\mathrm{adv},i}[u_i](x,t) + \mathbb{B}_{\mathrm{visc},i}[u_i](x,t).$$

where $\Phi[u] + \varphi(0) = u_i(0,t) \quad \forall\ i$ and $\sigma_i = \pm 1$ (arising from change of variables $x \mapsto -x$ on e_1).

- Showing that right-hand side of the above defines a contraction on a ball in $C_b\left([0,T]\,; C_b\left(X\right)\right)$ is straightforward since we use sup-norms and $\Phi, \mathbb{B}_{\mathrm{adv},i}, \mathbb{B}_{\mathrm{visc},i}$ are all integral ops. w/ rapidly decaying kernels.

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Proof of LWP (cont.)

- Contraction Mapping Thm. $\Rightarrow$ fixed-pt. form of the problem has a solution in $C_b(X)$, unique in some ball in $C_b\left([0, T]\,; C_b\left(X\right)\right)$.

- To show *unconditional uniqueness* (without imposing soln. lives in ball), use a bootstrap argument.

- The particular structure of $\Phi, \mathbb{B}_{\mathrm{adv},i}, \mathbb{B}_{\mathrm{visc},i}$ as nested integrals with nice kernels shows that the fixed point has the claimed differentiability ie. it gives a classical solution to gBBMB on each edge

- Similarly, we can get continuous (actually Lipschitz) dependence on the initial data $\varphi(x)$, possibly by shrinking the existence time T. Proof is now done!

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Energy Estimate

Definition

The **energy** *E of $u\colon X \to \mathbb{R}$ is defined to be*

$$\frac{1}{2}\|u\|^2_{H^1(X)} = \frac{1}{2}\sum_i \int_0^\infty |u_i|^2 + \mu_i^2\ |u_{i,x}|^2\ \mathrm{d}x.$$

(w/ change of vars. for u_1 implicit). $H^1(X) =$ functions in $C_b(X)$ with finite energy.

Proposition (Energy Estimate)

For a solution $u(x,t)$ to gBBMB on X,

$$\frac{\mathrm{d}E}{\mathrm{d}t} \leq -h^2(t)\ \left[\sum_i \sigma_i\left(\frac{\alpha_i}{2} + \frac{\gamma_i}{(p+1)(p+2)}h^p(t)\right)\right]$$

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Energy Method for GWP

The equation for $\mathrm{d}E/\mathrm{d}t$ gives us a lot of mileage in proving global well-posedness:

Corollary (Global Well-Posedness of Network Problem)

If

$$\sum_{i=1}^{N} \sigma_i \alpha_i \geq 0, \ \sum_{i=1}^{N} \sigma_i \gamma_i \geq 0$$

and p is even, then the solution to gBBMB valid up to time T can be extended to a unique global-in-time solution $u \in C_b\left([0,\infty); H^1(X)\right)$.

If $\sum_i \sigma_i \gamma_i = 0$, the demand of an even p can be relaxed.

Sketch of GWP Proof

- Using the energy estimate, we find that $\frac{\mathrm{d}E}{\mathrm{d}t} \leq 0$ by hypothesis. So the $H^1(X)$-norm of a solution to gBBMB on X decreases over time.

- By the Sobolev embedding $H^1(X) \hookrightarrow C_b(X)$, have that $C_b(X)$-norm of a soln. also decreases over time.

- Thus LWP result can be iterated: if T is small existence time, $u(x, T)$ can be used as "initial data" to get soln on $[T, T + T']$,...

- Energy argument + junction conditions give uniqueness. Proof is complete.

Discussion of GWP

Are these parameter restrictions interesting?

- $\alpha_i > 0$ is necessary for long linear waves to go towards $+\infty$, and $\gamma_i \geq 0$ by hypothesis
- According to Erbay et al. 1992, $p = 2$ may capture genuinely nonlinear behaviour in a wider variety of elastic materials compared to $p = 1$
- Since we expect coeffs. will not change too much from edge to edge, $\sum_{i=1}^{N} \sigma_i \alpha_i \geq 0$, $\sum_{i=1}^{N} \sigma_i \gamma_i \geq 0$ seems reasonable
- In the $N = 2$ case, get GWP if only μ_i, ν_i vary from edge to edge, which means only linear (visco)elasticity of arteries changes. This is fine for a toy model

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- According to Erbay et al. 1992, $p = 2$ may capture genuinely nonlinear behaviour in a wider variety of elastic materials compared to $p = 1$
- Since we expect coeffs. will not change too much from edge to edge, $\sum_{i=1}^{N} \sigma_i \alpha_i \geq 0$, $\sum_{i=1}^{N} \sigma_i \gamma_i \geq 0$ seems reasonable
- In the $N = 2$ case, get GWP if only μ_i, ν_i vary from edge to edge, which means only linear (visco)elasticity of arteries changes. This is fine for a toy model

Numerics Outline 1

- Solitary wave solns. to gBBM (blood pulses) parameterized by speed $c \in (\alpha, \infty)$ and initial peak location x_0:

$$u(x,t) = \left[A(p,\mu,\alpha,c) \cosh\left(\frac{x - x_0 - ct}{W(p,\mu,\alpha,c)} \right) \right]^{-2/p}.$$

- Main purpose of simulations here is to investigate how incoming solitary waves scatter off the junction of a two-edge network. Linear scattering can be understood analytically

- Simulations based on a **finite difference scheme** of Eilbeck and McGuire developed for BBM (three-level implicit, second order) and operator splitting to accommodate dissipative term.

Numerics Outline 2

- In most simulations, we increase linear elasticity μ from edge to edge. Physically: solitary blood pulse is going from a less rigid artery segment to a more rigid segment (representing sclerosis)

- Later, also change viscoelastic coefficient ν from edge to edge

- To measure performance, look at how well simulation conserves mass of the solution,

$$M(t) = \sum_i \int_{e_i} u(x,t).$$

Denote percent relative error in mass by

$$\delta M \doteq 100 \, \frac{|M(t) - M(0)|}{M(0)} \, \%.$$

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$\nu = 0$ Solitary Wave Scattering 1: No Reflection

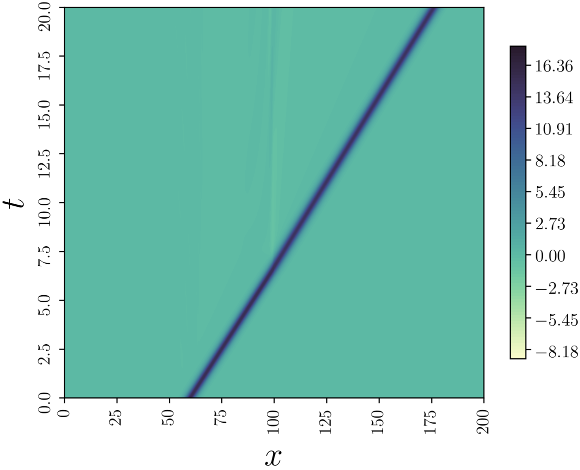

Figure: $u(x, t)$ for a BBM solitary wave, $\mu_2/\mu_1 = 1.1$. Barely any change when moving across the interface ($x = 100$). $\delta M \leq 0.5\%$ here.

$\nu = 0$ Solitary Wave Scattering 2: Reflection

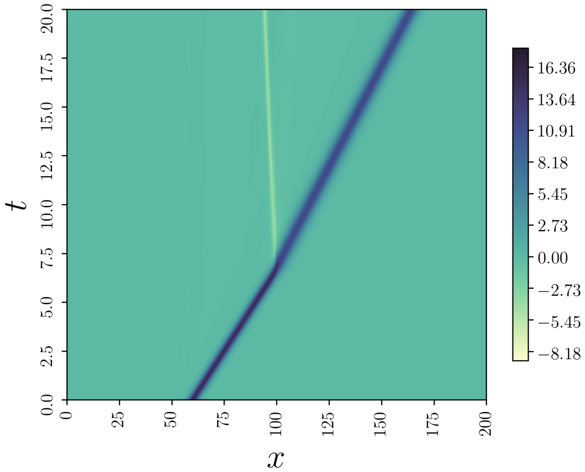

Figure: $u(x, t)$ for a BBM solitary wave, $\mu_2/\mu_1 = 1.5$. Small, slow solitary wave reflected at interface ($x = 100$). Transmitted solitary wave is slower and wider. $\delta M \leq 0.5\%$ here.

$\nu \neq 0$ Solitary Wave Scattering 1: $\mu_1 = \mu_2$

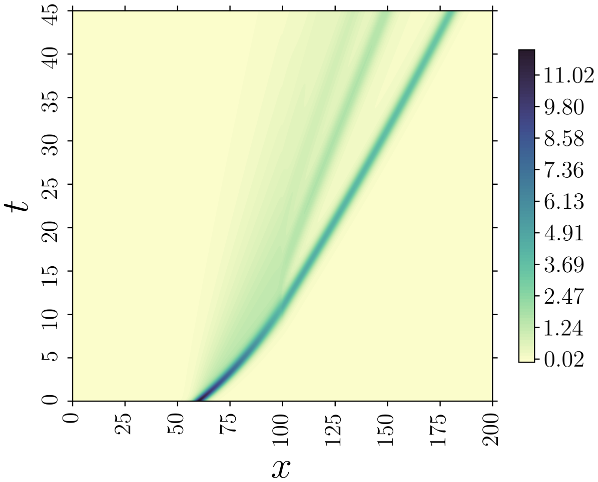

Figure: $\mu_1 = \mu_2 = 1, \nu_1 = 1, \nu_2 = 0.1$. Dissipative term creates a long "tail" in initial solitary wave. Wavemaker is generated at junction. $\delta M \leq 0.13\%$ for this test.

$\nu \neq 0$ Solitary Wave Scattering 2: $\mu_2/\mu_1 = 1.5$

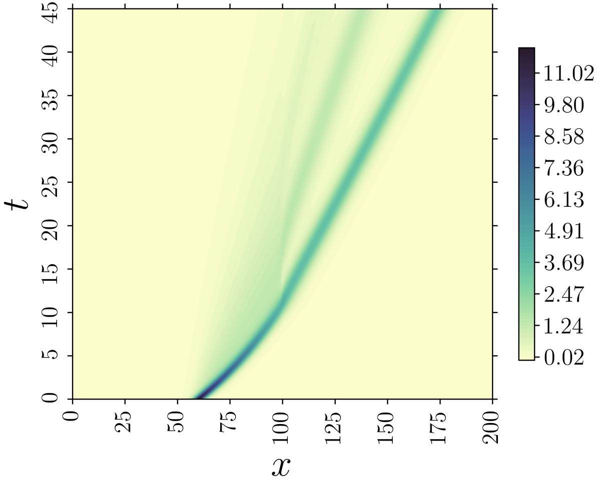

Figure: $\mu_1 = 1, \mu_2 = 1.5, \nu_1 = 1, \nu_2 = 0.1$. Only the speed of outgoing solitary waves changes from previous picture. Viscoelasticity has killed the reflected solitary wave. $\delta M \leq 0.2\%$ for this test.

Main Observations from Numerics (conjectures for analysis!)

- Speed of transmitted solitary wave changes across junction when μ varies.
- Reflection of solitary waves is possible when $\nu = 0$ and μ_2/μ_1 is sufficiently large (further experiments not shown here actually imply reflection is also *amplitude*-dependent!)
- Sufficiently large jump in viscoelasticity can lead to wavemaker generated on outgoing edge.
- Strong viscoelasticity kills reflected solitary wave.
- More sophisticated numerical method is needed to efficiently handle multiple network edges and make behaviour at computational boundaries more physical. *Work in progress*.

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Summary of Presentation

- Formulated gBBMB to guarantee mass conservation in full generality
- Proved local well-posedness, can often be extended to global
- Numerical simulations of nonlinear scattering with and without viscoelasticity lead to interesting questions

Future Plans 1

- While reading up on BBM and gBBMB, I did not find a great deal of recent work on these eqns. In particular, long-time asymptotics has not been completely understood
- Estimates from the 80's and 90's (mainly from Albert, Souganidis & Strauss, and Dziubanski & Karch) on asymptotics of gBBM, gBBMB for p large enough
- p can be made lower for gBBMB asymptotics, but dissipation swamps dispersion (see for instance Bona & Luo 2001)

Question: what do modern PDE/harmonic analysis tools have to tell us about the effects of dispersion on the long-time behaviour of gBBM? How large does p have to be?

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Future Plans 2

Initial avenues of investigation: look at recent work on generalized KdV, since BBM is a substitute for KdV

- Harrop-Griffiths 2016: asymptotics for small solns of mKdV ($u^2 u_x$ nonlinear term) without leveraging complete integrability. Primarily spatial methods (vector fields)

- Germain, Pusateri, & Rousset 2016: similar asymptotics, with applications to proving solitary wave stability. Primarily Fourier methods.

Hopefully the techniques in both these papers can allow us to take $p \geq 2$ in gBBM. I have not yet seen good results for p this small. BUT, stationary phase techniques do not appear 100% useful for gBBM: big trouble! Stay tuned.

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Questions?