Existence of parabolic minimizers to the total variation flow on metric measure spaces

Cintia Pacchiano Camacho

February 5th

Abstract

We give an existence proof for variational solutions u associated to the total variation flow. Here, the functions being considered are defined on a metric measure space $(\mathcal{X}, d, \mu)$. For such parabolic minimizers that coincide with a time-independent Cauchy-Dirichlet datum u_0 on the parabolic boundary of a space-time-cylinder $\Omega \times (0, T)$ with $\Omega \subset \mathcal{X}$ an open set and $T > 0$, we prove existence in the weak parabolic function space $L^1_w(0, T; \mathrm{BV}(\Omega))$. In this paper, we generalize results from a previous work by Bögelein, Duzaar and Marcellini and argue completely on a variational level. This is a join project with Vito Buffa and Michael Collins, from Friedrich-Alexander-Universität Erlangen-Nürnberg.

1 Motivation

Our aim is to show existence for parabolic minimizers to the total variation flow on metric measure spaces. More precisely, we consider minimizers of integral functionals that are related to scalar functions $u : \Omega \times (0, T) \to \mathbb{R}$ which satisfy the inequality

$$\iint u\partial_t \phi \mathrm{d}\mu \mathrm{d}t + \int \|Du(t)\| \mathrm{d}t \leq \int \|D(u + \varphi)(t)\| \mathrm{d}\mu, \tag{1}$$

where $\|Du\|$ denotes the total variation of u. Here, $\Omega \subset \mathcal{X}$ is a bounded domain, where $(\mathcal{X}, d, \mu)$ is a metric measure space with a metric d and a measure μ.

The Total Variation Flow (TVF) does not have any regularizing effects. Therefore, it is natural to expect the existence of solutions in the class of functions of bounded variation

(BV functions). As already mentioned, instead of the classical Euclidean setting, we intend to work in the general setting of metric measure spaces. During the past two decades, a theory of Sobolev and BV functions has been developed in this abstract setting. A central motivation for developing such a theory has been the desire to unify the assumptions and methods employed in various specific spaces, such as weighted Euclidean spaces, Riemannian manifolds, Heisenberg groups, graphs, etc.

In the setting of a metric measure space, the classical calculus known from the Euclidean space $\mathbb{R}^n$ is no longer available and instead of distributional derivatives, the space BV of functions with bounded variation has to be introduced by a relaxation approach [43], that makes use of the notion of upper gradients. An alternative approach to BV via derivations [12], is also presented. BV functions are a somewhat more general class than Sobolev functions, in the sense that they may have discontinuities and even "jumps", but are nonetheless differentiable in a very weak sense. The class has many applications, for example, as generalized solutions to partial differential equations with linear growth conditions, which often arise in the calculus of variations, physics, mechanics and image processing.

We deal with parabolic minimizers on parabolic cylinders $\Omega_T := \Omega \times (0,T)$ with $\Omega \subset \mathcal{X}$ bounded and open and $T > 0$. $\mathcal{X}$ denotes a metric measure space that fulfills a doubling property with respect to the metric d and the measure μ and supports a suitable Poincaré inequality. We generalize results which have recently been proven by Bögelein, Duzaar and Marcellini in [5], while we restrict ourselves to the simplest case where the functional in question depends only on the total variation itself.

2 Basic Definitions and Examples

2.1 Notations

Let $(\mathcal{X}, d, \mu)$ be a separable, connected metric measure space, i.e. $(\mathcal{X}, d)$ is a complete, separable and connected metric space endowed with a Borel measure μ on $\mathcal{X}$. The measure μ is assumed to fulfill a *doubling property*, i.e. there exists a constant $c \geq 1$, such that

$$0 < \mu\left(B_{2r}(x)\right) \leq c \cdot \mu\left(B_r(x)\right) < \infty \tag{2}$$

for all radii $r > 0$ and centres $x \in \mathcal{X}$. Here $B_r(x) \coloneqq \{y \in \mathcal{X} : d(x,y) < r\}$ denotes the open ball with radius r and centre x with respect to the metric d. The *doubling constant* is defined as

$$c_d \coloneqq \inf\{c \geq 1 : (2) \text{ holds true}\}. \tag{3}$$

A complete metric measure space that fulfills the doubling property is proper, meaning that all closed and bounded subsets are compact, [2, Proposition 3.1].

Following the concept of Heinonen and Koskela [26], we call a Borel function $g : \mathcal{X} \to [0, \infty]$ an *upper gradient* for an extended real-valued function $u : \mathcal{X} \to [-\infty, \infty]$ if for all $x, y \in \mathcal{X}$ and all rectifiable curves $\gamma : [0, L_\gamma] \to \mathcal{X}$ with $\gamma(0) = x, \gamma(L_\gamma) = y$ there holds

$$|u(x) - u(y)| \leq \int_\gamma g \, \mathrm{d}s. \tag{4}$$

Moreover, if a non-negative and measurable function g fulfills (4) for p-almost every curve as before, meaning that the family of curves for which (4) fails has p-modulus zero, then g is called *p-weak upper gradient*.
For $1 \leq p < \infty$ and a fixed open subset $\Omega \subset \mathcal{X}$ we define the vector space

$$\tilde{\mathbb{N}}^{1,p}(\Omega) := \{u \in L^p(\Omega) : \exists \text{ } p\text{-weak upper gradient } g \in L^p(\Omega) \text{ of } u\}.$$

$L^p(\Omega)$ denotes the usual Lebesgue space. The space $\tilde{\mathbb{N}}^{1,p}(\Omega)$ is endowed with the semi-norm

$$\|u\|_{\tilde{\mathbb{N}}^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \|g_u\|_{L^p(\Omega)}, \tag{5}$$

where g_u denotes the *minimal p-weak upper gradient* of u, i.e. $\|g_u\|_{L^p(\Omega)} = \inf \|g\|_{L^p(\Omega)}$, with the infimum being taken over all p-weak upper gradients of u. Introducing the equivalence relation

$$u \sim v \iff \|u - v\|_{\tilde{\mathbb{N}}^{1,p}(\Omega)} = 0,$$

we define the *Newtonian space* $\mathbb{N}^{1,p}(\Omega)$ as the quotient space

$$\mathbb{N}^{1,p}(\Omega) := \tilde{\mathbb{N}}^{1,p}(\Omega)/\sim,$$

which we endow with the quotient norm $\|\cdot\|_{\mathcal{N}^{1,p}(\Omega)}$ defined as in (5). Since this definition clearly depends on the metric d and the measure μ, we abuse the notation $\mathbb{N}^{1,p}(\Omega)$ as an abbrevation for $\mathbb{N}^{1,p}(\Omega, d, \mu)$. For more details on metric measure spaces we refer to [2, 28].
In addition to the doubling property, we demand that the metric measure space $(\mathcal{X}, d, \mu)$ supports a weak $(1,1)$-Poincaré inequality, in the sense that there exist a constant $c_P > 0$ and a dilatation factor $\tau \geq 1$ such that for all open balls $B_\varrho(x_0) \subset B_{\tau\varrho}(x_0) \subset \mathcal{X}$, for all L^1-functions u on $\mathcal{X}$ and all upper gradients $\tilde{g}_u$ of u there holds

$$\fint_{B_\varrho(x_0)} |u - u_{\varrho,x_0}| \, \mathrm{d}\mu \leq c_P \varrho \fint_{B_{\tau\varrho}(x_0)} \tilde{g}_u^p \, \mathrm{d}\mu, \tag{6}$$

where the symbol

$$u_{\varrho,x_0} := \fint_{B_\varrho(x_o)} u \, \mathrm{d}\mu := \frac{1}{\mu(B_\varrho(x_0))} \int_{B_\varrho(x_o)} u \, \mathrm{d}\mu$$

denotes the mean value integral of the function u on the ball $B_\varrho(x_0)$ with respect to the measure μ. Poincaré inequalities on metric measure spaces have been studied quite extensively in the literature, see for example [3, 4, 27, 29, 33, 35, 37, 44, 45].

Now, we recall the definition and some basic properties of functions of bounded variation, see [43]. For $u \in L^1_{\mathrm{loc}}(\mathcal{X})$, we define the total variation of u on $\mathcal{X}$ to be

$$\|Du\|(\mathcal{X}) \coloneqq \inf\left\{\liminf_{i\to\infty}\int_{\mathcal{X}} g_{u_i}\,\mathrm{d}\mu : u_i \in \mathrm{Lip}_{\mathrm{loc}}(\mathcal{X}),\ u_i \to u \text{ in } L^1_{\mathrm{loc}}(\mathcal{X})\right\},$$

where each g_{u_i} is the minimal 1-weak upper gradient of u_i. We say that a function $u \in L^1(\mathcal{X})$ is of *bounded variation*, by notation $u \in \mathrm{BV}(\mathcal{X})$, if $\|Du\|(\mathcal{X}) < \infty$. By replacing $\mathcal{X}$ with an open set $\Omega \subset \mathcal{X}$ in the definition of the total variation, we can define $\|Du\|(\Omega)$. The norm in BV is given by

$$\|u\|_{\mathrm{BV}(\Omega)} \coloneqq \|u\|_{L^1(\Omega)} + \|Du\|(\Omega).$$

It was shown in [43, Theorem 3.4] that for $u \in \mathrm{BV}(\mathcal{X})$, the total variation $\|Du\|$ is the restriction to the class of open sets of a finite Radon measure defined on the class of all subsets of $\mathcal{X}$. This outer measure is obtained from the map $\Omega \mapsto \|Du\|(\Omega)$ on open sets $\Omega \subset \mathcal{X}$ via the standard Carathéodory construction. Thus, for an arbitrary set $A \subset \mathcal{X}$ one can define

$$\|Du\|(A) \coloneqq \inf\left\{\|Du\|(\Omega) : \Omega \text{ open}, A \subset \Omega\right\}.$$

2.2 Parabolic function spaces

For a Banach space B and $T > 0$, the space

$$C^0([0,T];B)$$

consists of all continuous functions $u : [0,T] \to B$ satisfying

$$\|u\|_{C^0([0,T];B)} \coloneqq \max_{0\le t\le T} \|u(t)\|_B < \infty.$$

Naturally, for $\alpha \in (0,1]$, the space

$$C^{0,\alpha}([0,T];B)$$

consists of those functions $u \in C^0([0,T];B)$, for which additionally

$$\sup_{s,t\in[0,T]} \frac{\|u(s)-u(t)\|_B}{|s-t|^\alpha} < \infty$$

holds true.

In the Euclidean case, it can be shown via integration by parts that the space BV can be written as the dual space of a separable Banach space, see [1, Remark 3.12]. Since this tool is not available in the metric setting (at least not in the sense as it is understood in the Euclidean case), a different approach has to be taken.

2.2.1 The space BV via derivations

For the following definitions and properties, we followed [12], see also [7, 8]. Let $\mathrm{Lip}_{\mathrm{bs}}(\mathcal{X})$ denote the space of Lipschitz functions on $\mathcal{X}$ with bounded support and $L^0(\mathcal{X})$ the space of measurable functions. By a (Lipschitz) *derivation* we denote a linear map $\mathfrak{d} : \mathrm{Lip}_{\mathrm{bs}}(\mathcal{X}) \to L^0(\mathcal{X})$ such that the Leibniz rule

$$\mathfrak{d}(fg) = f\mathfrak{d}(g) + g\mathfrak{d}(f)$$

holds true for all $f, g \in \mathrm{Lip}_{\mathrm{bs}}(\mathcal{X})$ and for which there exists a function $h \in L^0(\mathcal{X})$ such that for μ-a.e. (almost every) $x \in \mathcal{X}$ and all $f \in \mathrm{Lip}_{\mathrm{bs}}(\mathcal{X})$ there holds

$$|\mathfrak{d}(f)|(x) \leq h(x) \cdot \mathrm{Lip}_a(f)(x), \tag{7}$$

where $\mathrm{Lip}_a(f)(x)$ denotes the asymptotic Lipschitz constant of f at x. The set of all such derivations will be denoted by $\mathrm{Der}(\mathcal{X})$. The smallest function h satisfying (7) will by denoted by $|\mathfrak{d}|$ and we are going to write $\mathfrak{d} \in L^p$ when we mean to say $|\mathfrak{d}| \in L^p$.

For given $\mathfrak{d} \in \mathrm{Der}(\mathcal{X})$ with $\mathfrak{d} \in L^1_{\mathrm{loc}}(\mathcal{X})$ we define the *divergence operator* $\nabla \cdot (\mathfrak{d}) : \mathrm{Lip}_{\mathrm{bs}}(\mathcal{X}) \to \mathbb{R}$ as

$$f \mapsto -\int_{\mathcal{X}} \mathfrak{d}(f)\,\mathrm{d}\mu.$$

We say that $\nabla \cdot (\mathfrak{d}) \in L^p(\mathcal{X})$ if this operator admits an integral representation via a unique L^p-function h, i.e.

$$\int_{\mathcal{X}} \mathfrak{d}(f)\,\mathrm{d}\mu = -\int_{\mathcal{X}} hf\,\mathrm{d}\mu.$$

For all $p, q \in [1, \infty]$ we shall set

$$\mathrm{Der}^p(\mathcal{X}) \coloneqq \{\mathfrak{d} \in \mathrm{Der}(\mathcal{X}) : \mathfrak{d} \in L^p(\mathcal{X})\}$$

and

$$\mathrm{Der}^{p,q}(\mathcal{X}) \coloneqq \{\mathfrak{d} \in \mathrm{Der}(\mathcal{X}) : \mathfrak{d} \in L^p(\mathcal{X}),\ \nabla \cdot (\mathfrak{d}) \in L^q(\mathcal{X})\}.$$

When $p = \infty = q$ we will write $\mathrm{Der}_b(\mathcal{X})$ instead of $\mathrm{Der}^{\infty,\infty}(\mathcal{X})$. The domain of the divergence is characterized as

$$D(\nabla\cdot) \coloneqq \{\mathfrak{d} \in \mathrm{Der}(\mathcal{X}) : |\mathfrak{d}|, \nabla\cdot(\mathfrak{d}) \in L^1_{\mathrm{loc}}(\mathcal{X})\}.$$

For $u \in L^1(\mathcal{X})$ we say that u is of *bounded variation (in the sense of derivations)* in $\mathcal{X}$, denoted $u \in \mathrm{BV}_{\mathfrak{d}}(\mathcal{X})$, if there is a linear and continuous map $L_u : \mathrm{Der}_b(\mathcal{X}) \to \mathbf{M}(\mathcal{X})$ such that

$$\int_{\mathcal{X}} \mathrm{d}L_u(\mathfrak{d}) = -\int_{\mathcal{X}} u\nabla\cdot(\mathfrak{d})\,\mathrm{d}\mu \tag{8}$$

for all $\mathfrak{d} \in \mathrm{Der}_b(\mathcal{X})$ and satisfying $L_u(h\mathfrak{d}) = hL_u(\mathfrak{d})$ for any bounded $h \in \mathrm{Lip}(\mathcal{X})$, where $\mathbf{M}(\mathcal{X})$ denotes the space of finite signed Radon measures on $\mathcal{X}$.

As observed in [12], the characterization of BV in the sense of derivations is well-posed. If we take any two maps L_u, $\tilde{L}_u$ as in (8), the Lipschitz-linearity of derivations ensures that $L_u(\mathfrak{d}) = \tilde{L}_u(\mathfrak{d})$ μ-a.e. for all $\mathfrak{d} \in \mathrm{Der}_b(\mathcal{X})$. The common value will be then denoted by $Du(\mathfrak{d})$.

From [12] we know that for $u \in \mathrm{BV}_{\mathfrak{d}}(\mathcal{X})$ there exists a non-negative, finite Radon measure $\nu \in \mathbf{M}(\mathcal{X})$ such that for every Borel set $B \subset \mathcal{X}$ one has

$$\int_B \mathrm{d}Du(\mathfrak{d}) \leq \int_B |\mathfrak{d}|^*\,\mathrm{d}\nu \tag{9}$$

for all $\mathfrak{d} \in \mathrm{Der}_b(\mathcal{X})$, where $|\mathfrak{d}|^*$ denotes the upper-semicontinuous envelope of $|\mathfrak{d}|$. The least measure ν satisfying (9) will be denoted by $\|Du\|_{\mathfrak{d}}$, the total variation of u (in the sense of derivations). Moreover, we have

$$\|Du\|_{\mathfrak{d}}(\mathcal{X}) = \sup\{|Du(\mathfrak{d})(\mathcal{X})| : \mathfrak{d} \in \mathrm{Der}_b(\mathcal{X}), |\mathfrak{d}| \leq 1\}.$$

Finally, by [12, Theorem 7.3.4], the classical representation formula for $\|Du\|_{\mathfrak{d}}$ holds, in the sense that if $\Omega \subset \mathcal{X}$ is any open set, then

$$\|Du\|_{\mathfrak{d}}(\Omega) = \sup\left\{\int_\Omega u\nabla\cdot(\mathfrak{d})\,\mathrm{d}\mu : \mathfrak{d} \in \mathrm{Der}_b(\mathcal{X}), \mathrm{supp}(\mathfrak{d}) \Subset \Omega, |\mathfrak{d}| \leq 1\right\}. \tag{10}$$

From [12, Theorem 7.3.7] we obtain that if $(\mathcal{X}, d, \mu)$ is a complete and separable metric measure space endowed with a locally finite measure μ (as in the case of this paper), then

$$\mathrm{BV}(\mathcal{X}) = \mathrm{BV}_{\mathfrak{d}}(\mathcal{X})$$

and in particular, the respective notions of the total variation coincide. Therefore, from now on, we are only going to write $\mathrm{BV}(\mathcal{X})$ and $\|Du\|$ without making any further distinction.

2.2.2 Weak parabolic function spaces

For $T > 0$ and an open subset $\Omega \subset \mathcal{X}$ we write Ω_T for the space-time cylinder $\Omega \times (0, T)$. For the concept of variational solutions we are going to make use of the space

$$L^1_w(0, T; \mathrm{BV}(\Omega)),$$

where the suffix w stands for 'weak'. This space consists of those $v \in L^1(\Omega_T)$, such that there holds:

- $v(\cdot, t) \in \mathrm{BV}(\Omega)$ for a.e. $t \in (0, T)$,
- $\int_0^T \|Dv(t)\|(\Omega)\mathrm{d}t < \infty$.
- The mapping $t \mapsto v(\cdot, t)$ is *weakly measurable*, i.e. the mapping

$$(0, T) \ni t \longmapsto \int_\Omega v(t)\nabla \cdot (\mathfrak{d})\, \mathrm{d}\mu \tag{11}$$

  is measurable for all $\mathfrak{d} \in \mathrm{Der}_b(\Omega)$ with $\mathrm{supp}(\mathfrak{d}) \Subset \Omega$.

Remark 1. *In the case of the gradient flow, i.e. a functional with p-growth for $p > 1$, the parabolic function spaces considered are usually $L^p(0, T; \mathrm{N}^{1,p}(\Omega))$, which consist of mappings $v : (0, T) \to \mathrm{N}^{1,p}(\Omega)$ that are strongly measurable in the sense of Bochner, see [28, Chapter 3]. In the case at hand, that is $p = 1$, one would consider the Bochner space $L^1(0, T; \mathrm{BV}(\Omega))$. But the strong measurability in the sense of Bochner is too restrictive, since many simple examples - like the space-time cone $u(t) = \mathbb{1}_{B_t(x_0)}$ - are not strongly measurable in the sense of Bochner, since their image is not separable in $\mathrm{BV}(\Omega)$. Therefore, the strong measurability condition is replaced with a weaker one.*
Note that the weak measurability of a function in $L^1_w(0, T; \mathrm{BV}(\Omega))$ is not to be confused with the weak measurability of a Banach space-valued function in the sense of Pettis' theorem, see again [28, Chapter 3].

Remark 2. *In the Euclidean case, i.e. $\mathcal{X} = \mathbb{R}^n$ for some $n \in \mathbb{N}$, the notion of weak measurability as in (11) is usually understood in the sense that the pairing*

$$(0, T) \ni t \mapsto \langle Dv(t), \varphi\rangle = -\int_\Omega v(t)\nabla \cdot (\varphi)\, \mathrm{d}x$$

is measurable for any $\varphi \in C^1_0(\Omega; \mathbb{R}^n)$.

Indeed, the approach by derivations as introduced before yields this classical notion of weak measurability. To understand this, define for any $\varphi \in C^1_0(\Omega; \mathbb{R}^n)$ the mapping

$$\mathfrak{d}_\varphi : \mathrm{Lip}_{bs}(\Omega) \ni f \mapsto \langle \varphi, Df\rangle.$$

By Rademacher's theorem, the gradient Df is defined almost everywhere on Ω for a Lipschitz function f. It is easy to check that $\mathfrak{d}_\varphi$ fulfills the Leibniz rule and the property (7) with $g(x) = |\varphi(x)|$ almost everywhere. By integration by parts, we find that for the divergence operator of $\mathfrak{d}_\varphi$ there holds

$$\nabla \cdot (\mathfrak{d}_\varphi) : f \mapsto - \int_\Omega \langle \varphi, Df \rangle \, \mathrm{d}x = \int_\Omega \nabla \cdot (\varphi) f \, \mathrm{d}x.$$

Hence, the divergence of $\mathfrak{d}_\varphi$ is represented by $\nabla \cdot (\varphi)$. Thus, the weak measurabilty in the sense of (11) yields the measurabilty of the mapping

$$(0, T) \ni t \mapsto \int_\Omega v(t) \nabla \cdot (\varphi) \, \mathrm{d}x.$$

In view of (10), the mapping $[0, T] \ni t \mapsto \|Dv(t)\|(\Omega)$ is measurable for $v \in L^1_w(0, T; \mathrm{BV}(\Omega))$.

Furthermore, the limit of a sequence of functions in $L^1_w(0, T; \mathrm{BV}(\Omega))$ with uniformly bounded total variation is again a $L^1_w(0, T; \mathrm{BV}(\Omega))$-function.

2.3 Variational solutions

In the Euclidean case, i.e. $\mathcal{X} = \mathbb{R}^n$, one might consider the Cauchy-Dirichlet problem

$$\begin{cases} \partial_t u - \nabla \cdot \left(\dfrac{Du}{|Du|} \right) = 0 & \text{in } \Omega_T, \\ \hfill u = u_0 & \text{on } \partial_{\mathrm{par}} \Omega_T, \end{cases} \tag{12}$$

where $\partial_{\mathrm{par}} \Omega_T := (\overline{\Omega} \times \{0\}) \cup (\partial \Omega \times (0, T))$ denotes the *parabolic boundary* of Ω_T and u_0 is some given boundary data.

When trying to define a concept of Cauchy-Dirichlet problems like (12) for $u \in L^1_w(0, T; \mathrm{BV}(\Omega))$ on metric measure spaces, one has to overcome several difficulties. Indeed, similarly to what already observed in [5, Section 1.2], we point out that also in our case boundary values of BV-functions are a delicate to manage, since the trace operator is not continuous with respect to the weak*-convergence in $\mathrm{BV}(\Omega)$ - see for instance [1, Def. 3.11] - and the pairing in (11). A suitable strategy to treat this issue is to consider a slightly larger domain Ω^* that compactly countains the bounded open set Ω and to assume that the datum u_0 is defined on Ω^*. The boundary condition $u = u_0$ on the lateral boundary $\partial \Omega \times (0, T)$ could then be interpreted by requiring that $u(\cdot, t) = u_0$ a.e. on $\Omega^* \setminus \Omega$ for all $t \in (0, T)$. Thus said, from now on boundary values shall be understood in the following sense:

given $u_0 \in \mathrm{BV}(\Omega^*)$, a function u belongs to $\mathrm{BV}_{u_0}(\Omega)$
if and only if $u \in \mathrm{BV}(\Omega^*)$ and $u = u_0$ a.e. on $\Omega^* \setminus \Omega$.

The condition on the lateral boundary has to be read in the sense that there holds $u(\cdot, t) \in \mathrm{BV}_{u_0}(\Omega)$ for a.e. $t \in (0, T)$.

On the other hand, we do not have the possibility to explain derivatives such as in (12). Therefore we cannot consider Cauchy-Dirichlet problems like this. However, by an idea of Lichnewsky and Temam (see [38]), one can define the concept of *variational solutions*. Since this concept for solutions to a Cauchy-Dirichlet problem is described purely on a variational level, it can be extended to the concept of metric measure spaces.

To be precise, we assume Ω to be open and bounded, Ω^* open and bounded with $\Omega \Subset \Omega^*$ and

$$u_0 \in L^2(\Omega^*) \cap \mathrm{BV}(\Omega^*). \tag{13}$$

Where it makes sense, we are going to abbreviate $v(t) := v(\cdot, t)$.

Definition 3. Assume that the Cauchy-Dirichlet datum u_0 fulfills (13). A map $u : \Omega_T^* \to \mathbb{R}$, $T \in (0, \infty)$ in the class

$$L^1_w\left(0, T; \mathrm{BV}_{u_0}(\Omega)\right) \cap C^0\left([0, T]; L^2(\Omega^*)\right)$$

will be referred to as a *variational solution* on Ω_T to the Cauchy-Dirichlet problem for the total variation flow if and only if the variational inequality

$$\begin{aligned}\int_0^T \|Du(t)\|(\Omega^*)\,\mathrm{d}t \le \int_0^T &\left[\int_{\Omega^*} \partial_t v(v-u)\,\mathrm{d}\mu + \|Dv(t)\|(\Omega^*)\right]\mathrm{d}t \\ &- \frac{1}{2}\|(v-u)(T)\|^2_{L^2(\Omega^*)} + \frac{1}{2}\|v(0) - u_0\|^2_{L^2(\Omega^*)}\end{aligned} \tag{14}$$

holds true for any $v \in L^1_w\left(0, T; \mathrm{BV}_{u_0}(\Omega)\right)$ with $\partial_t v \in L^2(\Omega_T^*)$ and $v(0) \in L^2(\Omega^*)$. A map $u : \Omega^*_\infty \to \mathbb{R}$ is termed a *global variational solution* if

$$u \in L^1_w\left(0, T; \mathrm{BV}_{u_0}(\Omega)\right) \cap C^0\left([0, T]; L^2(\Omega^*)\right) \text{ for any } T > 0$$

and u is a variational solution on Ω_T for any $T \in (0, \infty)$.

Note that the time indepenent extension $v(\cdot, t) := u_0$ is an admissible comparison map in (14). Therefore, we have that

$$\int_0^T \|Du(t)\|(\Omega^*)\,\mathrm{d}t < \infty$$

for any variational solution u.

3 Results and proofs

Our main results concern the existence, uniqueness and regularity of variational solutions as follows:

Theorem 4. *Suppose that the Cauchy-Dirichlet datum u_0 fulfills the requirements of (13). Then, there exists a unique global variation solution in the sense of Definition 3.*

Theorem 5. *Suppose that the Cauchy-Dirichlet datum u_0 fulfills the requirements of (13). Then, any variational solution in the sense of Definition 3 on Ω_T with $T \in (0, \infty]$ satisfies*

$$\partial_t u \in L^2(\Omega^*) \text{ and } u \in C^{0,\frac{1}{2}}\left([0,\tau]; L^2(\Omega^*)\right) \text{ for all } \tau \in \mathbb{R} \cap (0, T].$$

Furthermore, for the time derivative $\partial_t u$ there holds the quantitative bound

$$\int_0^T \int_{\Omega^*} |\partial_t u|^2 \, \mathrm{d}\mu \, \mathrm{d}t \leq \|Du_0\|(\Omega^*).$$

Finally, for any $t_1, t_2 \in \mathbb{R}$ with $0 \leq t_1 < t_2 \leq T$ one has the energy estimate

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \|Du(t)\|(\Omega^*) \, \mathrm{d}t \leq \|Du_0\|(\Omega^*). \tag{15}$$

Our method of proof is aligned to the one proposed in the work of Bögelein, Duzaar and Marcellini [5]. For mappings $v : \Omega^* \times (0, \infty) \to \mathbb{R}$ and time-independent Cauchy-Dirichlet data $u_0 : \Omega^* \to \mathbb{R}$ such that the condition $v = u_0$ is fulfilled on the parabolic boundary of $\Omega \times (0, \infty)$ in the sense that $v(t) = u_0$ holds μ-almost everywhere on $\Omega^* \setminus \Omega$ for almost every $t \in (0, T)$, we consider the relaxed convex functionals

$$\mathcal{F}_\varepsilon[v] := \int_0^T e^{-\frac{t}{\varepsilon}} \left[\frac{1}{2} \int_{\Omega^*} |\partial_t v|^2 \, \mathrm{d}\mu + \frac{1}{\varepsilon} \|Dv(t)\|(\Omega^*)\right] \mathrm{d}t$$

for $\varepsilon \in (0, 1]$. The properties of the total variation allow the application of standard methods in the calculus of variations to ensure the existence of minimizers u_ε of $\mathcal{F}_\varepsilon$. To prove the existence of the minimizers u_ε of the relaxed convex functionals $\mathcal{F}_\varepsilon$ in our setting, we apply a compactness result by Simon [48]. In particular, this is used to identify the limit of a sequence of functions in $L^1_w(0, T; \mathrm{BV}(\Omega^*))$ since there are no standard compactness theorems that can be applied to this space. Now, these minimizers are expected to converge to parabolic minimizers as in (1) based on an idea in the Euclidean case. There, the authors compute the corresponding Euler-Lagrange-equation for $\mathcal{F}_\varepsilon$, see [5] for details. In order to establish this convergence, we argue completely on the level of minimizers. To this end, we follow an idea of Lichnewsky & Temam (see [38]) and thus introduce the concept of evolutionary variational solutions similarly to [5]. Precisely (see also Definition 3), we are looking at continuous mappings $u : (0,T) \to L^2(\Omega^*)$ that are also in the parabolic space $L^1_w(0,T;\mathrm{BV}_{u_0}(\Omega))$ and fulfill the variational inequality

$$\int_0^T \|Du(t)\|(\Omega^*)\,\mathrm{d}t \le \int_0^T \left[\int_{\Omega^*} \partial_t v(v-u)\,\mathrm{d}\mu + \|Dv(t)\|(\Omega^*)\right]\mathrm{d}t \\ - \frac{1}{2}\|(v-u)(T)\|^2_{L^2(\Omega^*)} + \frac{1}{2}\|v(0)-u_0\|^2_{L^2(\Omega^*)}.$$

By introducing a mollification in time in order to establish the existence of a L^2-time derivative, energy estimates are shown which lead to the convergence of the minimizers u_ε of $\mathcal{F}_\varepsilon$ to the variational solution u. Finally, it can be shown that these variational solutions are actually parabolic minimizers.

4 Brief Summary of Talk

1. Motivation. Why the study in metric measure spaces?
2. Basic Definitions for analysis in the general metric setting.
3. Parabolic function spaces. Understanding the function space we are in.
4. Proof of existence of a variational solution for the total variation flow.

References

[1] Ambrosio, L., Fusco, N., Pallara, D.: *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs. The Clarendon Press. Oxford University Press, New York, 2000.

[2] Björn, A., Björn J.: *Nonlinear Potential Theory on Metric Spaces*, EMS tracts in mathematics, Eur. Math. Soc., Zurich, 2011.

[3] Björn, J.: *Poincaré inequalities for powers and products of admissible weights*, Ann. Acad. Sci. Fenn. Math. **26** (2002), 175 – 188.

[4] Björn, J., Shanmugalingam, N.: *Poincaré inequalities, uniform domains and extension properties for Newton-Sobolev functions in metric spaces*, J. Math. Anal. Appl. **332** (2007), 190 – 208.

[5] Bögelein, V., Duzaar, F., Marcellini, P.: *A Time Dependent Variational Approach to Image Restoration*, SIAM J. Imaging Sci. (1) **8** (2015), 968 – 1006.

[6] Bögelein, V., Duzaar, F., Marcellini, P.: *Parabolic systems with p, q-growth: a variational approach*, Arch. Ration. Mech. Anal. (1) **210** (2013), 219 – 267.

[7] Buffa, V.: *BV Functions in Metric Measure Spaces: Traces and Integration by Parts Formulæ*, Ph.D. Thesis, Università degli Studi di Ferrara, 2017, available at: http://cvgmt.sns.it/paper/4110.

[8] Buffa, V., Comi, G.E., Miranda, M. Jr.: *On BV functions and essentially bounded divergence-measure fields in metric spaces*, preprint, available at: https://arxiv.org/abs/1906.07432.

[9] Cheeger, J.: *Differentialbility of Lipschitz functions on metric measure spaces*, Geometric And Functional Analysis **9** (1999), 428 – 517.

[10] Collins, M.: *Existence of variational solutions to a Cauchy-Dirichlet problem with time-dependent boundary data on metric measure spaces*, preprint – submitted for publication.

[11] Collins, M., Herán, A.: *Existence of parabolic minimizers on metric measure spaces*, Nonlinear Analysis **176** (2018), 56 – 83.

[12] Di Marino, S.: *Recent advances on BV and Sobolev Spaces in metric measure spaces*, Ph.D. Thesis, Scuola Normale Superiore, Pisa, 2014, available at: http://cvgmt.sns.it/paper/2568/.

[13] Ekeland, I., Temam, R.: *Convex Analysis and Variational Problems*, North-Holland, Amsterdam, 1976.

[14] Federer, H.: *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153 Springer-Verlag New York Inc., New York 1969. xiv+676 pp.

[15] Franchi, P., Hajlaz, P., Koskela, P.: *Definitions on Sobolev classes in metric spaces*, Annales de l'institut fourier (Grenoble) **49** (1999), 1903 – 1924.

[16] Fujishima, Y., Habermann, J.: *Global higher integrability for parabolic quasiminimizers in metric spaces*, Advances in Calculus of Variations (3) **10** (2017), 267 – 301.

[17] Fujishima, Y., Habermann, J.: *The stability problem for parabolic quasiminimizers in metric measure spaces*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl., XXIX (2018), 343 – 376.

[18] Fujishima, Y., Habermann, J., Kinnunen, J., Masson, M.: *Stability for parabolic quasi-minimizers*, Potential Analysis **41** (2014), 983 – 1004.

[19] Giusti, E.: *Minimal surfaces and functions of bounded variations*, Monographs in Mathematics, 80. Birkhäuser Verlag, Basel, 1984. xii+240 pp.

[20] Grigor'yan, A. A.: *The heat equation on noncompact Riemannian manifolds*, Mat. Sb. **182** (1992), 55 – 87. Translation in *Math. USSR-Sb.* **72** (1992), 47 – 77.

[21] Habermann, J.: *Higher integrability for vector-valued parabolic quasi-minimizers on metric measure spaces*, Ark. Mat. (1) **54** (2016), 85 – 123.

[22] Hajłasz, P.: *Sobolev spaces on an arbitrary metric space*, Potential Anal. **5** (1996), 403 – 415.

[23] Hajłasz, P.: *Sobolev spaces on metric measure spaces. Heat kernels and analysis on manifolds, graphs and metric spaces*, *in* Contemp. Math. **338**, pp. 173 – 218, Amer. Math. Soc., Providence, RI, 2003.

[24] Hajłasz, P., Koskela, P.: *Sobolev met Poincaré*, Mem. Amer. Math. Soc. **145** (2000), x + 101 pp.

[25] Hakkarainen, H., Kinnunen, J.: *The* BV*-capacity in metric spaces*, Manuscripta Math., 132 (2010), 51 – 73.

[26] Heinonen, J., Koskela, P.: *Quasiconformal maps in metric spaces with controlled geometry*, Acta Math. **181** (1998), 1–61.

[27] Heinonen, J., Koskela, P.: *A note on Lipschitz functions, upper gradients, and the Poincaré inequality*, New Zealand J. Math. **28** (1999), 37 – 42.

[28] Heinonen, J., Koskela, P., Shanmugalingam, N., Tyson, J. : *Sobolev Spaces on Metric Measure Spaces. An Approach Based on Upper Gradients*, New Mathematical Monographs, 27, Cambridge University Press, Cambridge, 2015.

[29] Keith, S., Zhong, X.: *The Poincaré inequality is an open ended condition*, Ann. of Math. (2) **167** (2008), 575 – 599.

[30] Kilpeläinen, T., Kinnunen, J., Martio, O.: *Sobolev spaces with zero boundary values on metric spaces*, Potential Anal. **12** (2000), 233 – 247.

[31] Kinnunen, J., Lindqvist, P.: *Pointwise behaviour of semicontinuous supersolutions to a quasilinear parabolic equation*, Ann. Mat. Pura Appl. (4) **185** (2006), 411 – 435

[32] Kinnunen, J., Marola, N., Miranda Jr., M., Paronetto, F.: *Harnack's inequality for parabolic De Giorgi classes in metric spaces*, Adv. Differential Equations **17** (2012), 801 – 832.

[33] Kinnunen, J., Martio, O.: *The Sobolev capacity on metric spaces*, Ann. Acad. Sci. Fenn. Math. **21** (1996), 367 – 382.

[34] Korte, R., Lahti, P., Li, X., Shanmugalingam, N.: *Notions of Dirichlet problem for functions of least gradient in metric measure spaces*, Rev. Mat. Iberoam. (6) **35** (2019), 1603 – 1648.

[35] Koskela, P., Shanmugalingam, N., Tuominen, H.: *Removable sets for the Poincaré inequality on metric spaces*, Indiana Univ. Math. J. **49** (2000), 333 – 352.

[36] Kronz, M.: *Some function spaces and spaces of homogeneous type*, Manuscripta Math. **106** (2001), 219 – 248.

[37] Laakso, T.: *Ahlfors Q-regular spaces with arbitrary Q ¿ 1 admitting weak Poincaré inequality*, Geom. Funct. Anal. **10** (2000), 111 – 123.

[38] Lichnewsky, A., Temam, R.: *Pseudosolutions of the time-dependent minimal surface problem*, J. Differential Equations (3) **30** (1978), 340 – 364.

[39] Maasalo, O. E., Zatorska-Goldstein, A.: *Stability of quasiminimizers of the p-Dirichlet integral with varying p on metric spaces*, J. Lond. Math. Soc. (2) **77**, no. 3 (2008), 771 – 788.

[40] Marola, N., Masson, M.: *On the Harnack inequality for parabolic minimizers in metric measure spaces*, Tohoku Math. J. (2) **65** (2013), 569 – 589.

[41] Masson, M., Miranda, M., Jr., Paronetto, F., Parviainen, M.: *Local higher integrability for parabolic quasiminimizers in metric spaces*, Ric. Mat. (2) **62** (2013), 279 – 305.

[42] Masson, M., Siljander, J.: *Hölder regularity for parabolic De Giorgi classes in metric measure spaces*, Manuscripta Mathematica (1) **142** (2013), 187 – 214.

[43] Miranda, M. Jr.: *Functions of bounded variation on "good" metric spaces*, J. Math. Pures Appl. (9) **82** (2003), no. 8, 975 – 1004.

[44] Saloff-Coste, L.: *A note on Poincaré, Sobolev and Harnack inequalities*, Int. Math. Res. Not. IMRN **2** (1992), 27 – 38.

[45] Saloff-Coste, L.: *Aspects of Sobolev-Type Inequalites*, London Mathematical Society Lecture Note Series **289**, Cambridge University Press, Cambridge, 2002.

[46] Shanmugalingam, N.: *Newtonian spaces: an extension of Sobolev spaces to metric measure spaces*, Revista Matemática Iberoamericana **16**(2) (2000), 243 – 279

[47] Shanmugalingam, N.: *Harmonic functions on metric spaces*, Illinois J. Math. **45** (2001), 1021 – 1050.

[48] Simon, J.: *Compact sets in the space $L^p(0,T;B)$*, Ann. Mat. Pura Appl. (4) **146** (1987), 65 – 96.

[49] Ziemer, W.P.: *Weakly differentiable functions. Sobolev spaces and functions of bounded variation*, Graduate Texts in Mathematics, 120. Springer-Verlag, New York, 1989.