Scattering of Small Solutions to the Generalized Benjamin-Bona-Mahony Equation

Adam Morgan

University of Toronto

March 19, 2021

Presentation Overview

1. Problem statement, intuition for the main theorem (scattering for GBBM)
2. Properties of the linear BBM
3. Rigorous proof of the main theorem, built up over several intermediate steps
4. Ideas for new approaches to long-time asymptotics?

Presentation Overview

1. Problem statement, intuition for the main theorem (scattering for GBBM)
2. Properties of the linear BBM
3. Rigorous proof of the main theorem, built up over several intermediate steps
4. Ideas for new approaches to long-time asymptotics?

Presentation Overview

1. Problem statement, intuition for the main theorem (scattering for GBBM)
2. Properties of the linear BBM
3. Rigorous proof of the main theorem, built up over several intermediate steps
4. Ideas for new approaches to long-time asymptotics?

Presentation Overview

1. Problem statement, intuition for the main theorem (scattering for GBBM)
2. Properties of the linear BBM
3. Rigorous proof of the main theorem, built up over several intermediate steps
4. Ideas for new approaches to long-time asymptotics?

Problem Statement + GWP

- We study the Cauchy problem for the **generalized Benjamin-Bona-Mahony equation (GBBM)**:

$$\begin{cases} u_t - u_{xxt} + u_x + u^p u_x = 0 \quad \forall\ (t,x) \in \mathbb{R} \times \mathbb{R} \\ u\big|_{t=0}(x) = u_0(x) \quad \forall\ x \in \mathbb{R}. \end{cases}$$

- This models long waves propagating in water, or in elastic blood vessels.
- Global well-posedness of this problem in

$$C_x^2 \cap H_x^1$$

is easy, thanks to **energy conservation**:

$$\|u(x,t)\|_{H_x^1} = \|u_0(x)\|_{H_x^1}.$$

Problem Statement + GWP

- We study the Cauchy problem for the **generalized Benjamin-Bona-Mahony equation (GBBM)**:

$$\begin{cases} u_t - u_{xxt} + u_x + u^p u_x = 0 \quad \forall\ (t,x) \in \mathbb{R} \times \mathbb{R} \\ u|_{t=0}(x) = u_0(x) \quad \forall\ x \in \mathbb{R}. \end{cases}$$

- This models long waves propagating in water, or in elastic blood vessels.
- Global well-posedness of this problem in

$$C_x^2 \cap H_x^1$$

is easy, thanks to **energy conservation**:

$$\|u(x,t)\|_{H_x^1} = \|u_0(x)\|_{H_x^1}.$$

Intuition for Scattering

- Energy conservation $\Rightarrow$ if $u_0(x)$ is small then $u(x,t)$ should always remain small.
- In turn, for p large enough, we should have

$$u^p u_x \approx 0,$$

which means the PDE is nearly equal to **linearized BBM (LBBM)**:

$$u_t - u_{xxt} + u_x = 0$$

- So, possibly after waiting some time for dispersion to tame any problems with the nonlinearity (more on this later), we can expect solns. to GBBM to act like solns. to LBBM.
- We then say that small solutions to GBBM with $p \gg 1$ **scatter**, at least from an intuitive point of view

Intuition for Scattering

- Energy conservation $\Rightarrow$ if $u_0(x)$ is small then $u(x, t)$ should always remain small.
- In turn, for p large enough, we should have

$$u^p u_x \approx 0,$$

which means the PDE is nearly equal to **linearized BBM (LBBM)**:

$$u_t - u_{xxt} + u_x = 0$$

- So, possibly after waiting some time for dispersion to tame any problems with the nonlinearity (more on this later), we can expect solns. to GBBM to act like solns. to LBBM.
- We then say that small solutions to GBBM with $p \gg 1$ **scatter**, at least from an intuitive point of view

Main Theorem

Theorem (Scattering in H^1, Dziubański & Karch '96)

Suppose $s \geq 7/2$ and $p > 4$. Let $u(x,t)$ denote the solution to GBBM with initial state $u_0(x)$.

Then, we can find $0 < \delta \ll 1$ such that

$$\|u_0\|_{L^1_x} + \|u_0\|_{H^s_x} < \delta$$

implies there exist functions $u_\pm(x,t) \in C^1_t\left(\mathbb{R}; H^s_x\right)$ satisfying the following:

1. *$u_\pm$ both provide classical solutions to LBBM and*
2. $\lim_{t\to\pm\infty} \|u_\pm(x,t) - u(x,t)\|_{H^1_x} = 0.$

The Road to the Main Thm.

- To say nonlinear solns. resemble linear ones, we should probably make sure we have a good idea what linear solutions look like
- Use the Fourier transform and asymptotics for oscillatory integrals (stationary phase method) to get basic information on LBBM solns.
- Need to prove a **dispersive estimate**
- Then, we show small solns. to the nonlinear problem also satisfy dispersive estimate: this is enough to prove the main thm.

The Road to the Main Thm.

- To say nonlinear solns. resemble linear ones, we should probably make sure we have a good idea what linear solutions look like
- Use the Fourier transform and asymptotics for oscillatory integrals (stationary phase method) to get basic information on LBBM solns.
- Need to prove a **dispersive estimate**
- Then, we show small solns. to the nonlinear problem also satisfy dispersive estimate: this is enough to prove the main thm.

Basics of LBBM 1

- Focus for now on IVP for LBBM:

$$\begin{cases} u_t - u_{xxt} + u_x = 0 & \forall\ (t,x) \in \mathbb{R} \times \mathbb{R} \\ u\big|_{t=0}(x) = u_0(x) & \forall\ x \in \mathbb{R}. \end{cases}$$

- Define an (invertible!) elliptic operator $M = 1 - \partial_x^2$, then LBBM can be written as

$$u_t = -M^{-1}\partial_x (u).$$

- The symbol of $\frac{1}{i} M^{-1}\partial_x$ (physically, the temporal frequency) is given by the **dispersion relation**

$$\omega(\xi) = \frac{\xi}{\langle \xi \rangle^2} \quad \left(\langle \xi \rangle = \sqrt{1+\xi^2} \right)$$

Think of this as writing temporal freq. as function of spatial freq. ξ (AKA "wavenumber")

Basics of LBBM 1

- Focus for now on IVP for LBBM:

$$\begin{cases} u_t - u_{xxt} + u_x = 0 \quad \forall\, (t,x) \in \mathbb{R} \times \mathbb{R} \\ u\big|_{t=0}(x) = u_0(x) \quad \forall\, x \in \mathbb{R}. \end{cases}$$

- Define an (invertible!) elliptic operator $M = 1 - \partial_x^2$, then LBBM can be written as

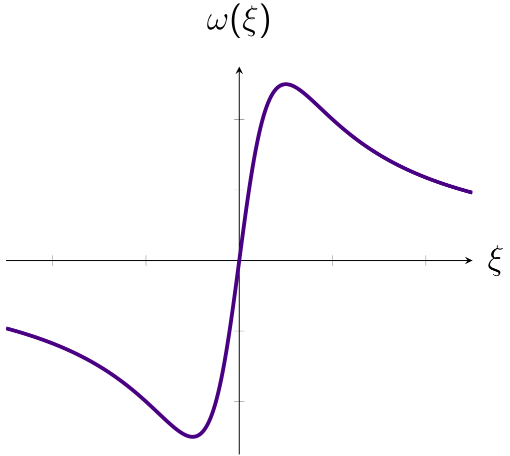
$$u_t = -M^{-1}\partial_x\,(u)\,.$$

- The symbol of $\frac{1}{i}M^{-1}\partial_x$ (physically, the temporal frequency) is given by the **dispersion relation**

$$\omega(\xi) = \frac{\xi}{\langle \xi \rangle^2} \quad \left(\langle \xi \rangle = \sqrt{1+\xi^2}\right)$$

  Think of this as writing temporal freq. as function of spatial freq. ξ (AKA "wavenumber")

Basics of LBBM 2



Basics of LBBM 3

- Using the Fourier transform, we can write soln. to LBBM as

$$e^{tM^{-1}\partial_x} u_0 \doteq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(\xi x - \omega(\xi) t)} \ \widehat{u_0}(\xi) \, \mathrm{d}\xi.$$

- Think of this as a weighted sum of normal modes (sinusoidal waves)
- When u_0 is Schwartz, soln. can also be pictured as an approximately localized (on a certain time scale) wavepacket.

LBBM Asymptotics 1

- Study $u(x,t)$ on spacetime rays $\Gamma_c = \{x = ct\}$. Given a fixed ray slope c, define the **LBBM phase** by

$$\phi(\xi) = c\xi - \omega(\xi)$$

so along Γ_c write

$$u(x,t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{i\phi(\xi)t} \, \widehat{u_0}(\xi) \, \mathrm{d}\xi.$$

- **Q**: How does $u(x,t)$ behave as $t \to \infty$?

LBBM Asymptotics 1

- Study $u(x,t)$ on spacetime rays $\Gamma_c = \{x = ct\}$. Given a fixed ray slope c, define the **LBBM phase** by

$$\phi(\xi) = c\xi - \omega(\xi)$$

  so along Γ_c write

$$u(x,t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{i\phi(\xi)t} \ \widehat{u_0}(\xi) \ \mathrm{d}\xi.$$

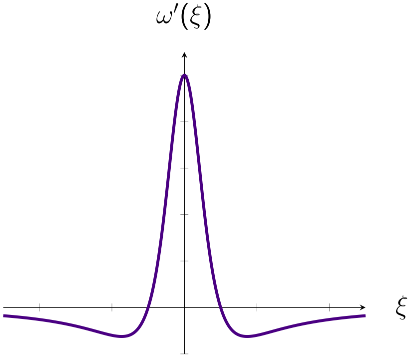
- **Q**: How does $u(x,t)$ behave as $t \to \infty$?

LBBM Asymptotics 2

- **A**: Use that the integrand oscillates!
- Dominant contribution to $u(x,t)$ along Γ_c comes at ξ for which $\phi'(\xi) = 0$: integrand oscillation is slowest here.
- Thus we look for ξ_0 such that $c = \omega'(\xi_0)$. Between 0 and 4 ξ_0 per c:

LBBM Asymptotics 2

- **A**: Use that the integrand oscillates!
- Dominant contribution to $u(x,t)$ along Γ_c comes at ξ for which $\phi'(\xi) = 0$: integrand oscillation is slowest here.
- Thus we look for ξ_0 such that $c = \omega'(\xi_0)$. Between 0 and 4 ξ_0 per c:



LBBM Asymptotics 3

- So: suppose $\widehat{u_0}$ is localized around ξ_0, then $e^{tM^{-1}\partial_x}u_0$ remains mostly localized in spacetime along $\Gamma_{\omega'(\xi_0)}$.
- Hence we can say the "velocity" of the wavepacket $e^{tM^{-1}\partial_x}u_0$ is $\omega'(\xi_0) =$ **group velocity**.
- If $\widehat{u_0}$ is more spread out, all of its component normal modes have different group vel., meaning wave packet "disperses" into a bunch of separated normal mode pieces as $t \to \infty$.
- Dispersion can help counteract nonlinear steepening.

LBBM Asymptotics 3

- So: suppose $\widehat{u_0}$ is localized around ξ_0, then $e^{tM^{-1}\partial_x}u_0$ remains mostly localized in spacetime along $\Gamma_{\omega'(\xi_0)}$.
- Hence we can say the "velocity" of the wavepacket $e^{tM^{-1}\partial_x}u_0$ is $\omega'(\xi_0) =$ **group velocity**.
- If $\widehat{u_0}$ is more spread out, all of its component normal modes have different group vel., meaning wave packet "disperses" into a bunch of separated normal mode pieces as $t \to \infty$.
- Dispersion can help counteract nonlinear steepening.

Stationary Phase Method

- Need the following approximation method to get more quantitative info on dispersion...

Theorem (Stationary Phase Estimate)

Suppose $\phi(\xi)$ is smooth, ξ_0 is the only zero of $\phi'(\xi)$, and there exists a natural number N such that

$$\phi^{(n)}(\xi_0) = 0 \text{ for } n = 1, 2, 3, ..., N-1.$$

Next, suppose that $f : \mathbb{R} \to \mathbb{C}$ is smooth and compactly supported. Then, for $t \gg 1$,

$$\int_{-\infty}^{\infty} f(\xi) e^{i\phi(\xi)t} \, \mathrm{d}\xi \approx C(\xi_0) \; f(\xi_0) \; e^{i\phi(\xi_0)t} \; t^{-\frac{1}{N}}$$

LBBM Asymptotics 4 + Dispersive Est.

- Since mass is conserved, wave dispersion implies the amplitude of a wavepacket decreases over time.
- Stationary phase estimate gives

$$\left\| e^{tM^{-1}\partial_x} u_0 \right\|_{L^\infty} \lesssim \|u_0\|_{L^1} \; t^{-1/3}, \quad t \gg 1,$$

- Above can be refined rigorously:

Proposition (LBBM Dispersive Estimate, Albert '89, Dziubański-Karch '96)

For any $s \geq \frac{7}{2}$, we have

$$\left\| e^{tM^{-1}\partial_x} u_0 \right\|_{L^\infty} \lesssim \left(\|u_0\|_{L^1} + \|u_0\|_{H^s}\right) \; \langle t \rangle^{-1/3}, \quad \forall \; t > 0.$$

LBBM Asymptotics 4 + Dispersive Est.

- Since mass is conserved, wave dispersion implies the amplitude of a wavepacket decreases over time.
- Stationary phase estimate gives

$$\left\| e^{tM^{-1}\partial_x} u_0 \right\|_{L^\infty} \lesssim \|u_0\|_{L^1} \; t^{-1/3}, \quad t \gg 1,$$

- Above can be refined rigorously:

Proposition (LBBM Dispersive Estimate, Albert '89, Dziubański-Karch '96)

For any $s \geq \frac{7}{2}$, we have

$$\left\| e^{tM^{-1}\partial_x} u_0 \right\|_{L^\infty} \lesssim \left(\|u_0\|_{L^1} + \|u_0\|_{H^s} \right) \; \langle t \rangle^{-1/3}, \quad \forall \; t > 0.$$

Discussion of Dispersive Est.

- Why do we need a Sobolev norm in dispersive est.? Recall: group velocity of a wavepacket tiny for $|\xi| \gg 1$
- So: if initial state consists of a high-frequency wave spatially localized around the origin, then even after a long time we will still see a high-frequency wave spatially localized near the origin.
- Thus any estimate on L^∞_x-norm of linear soln should depend on a norm that weighs high frequencies heavily: a Sobolev norm is built to do just this, and $s = 7/2$ is the "magic number"

Dispersive Est.: Proof Sketch 1

- One can determine a critical frequency magnitude $n_0 \approx 2$ based on stationary points of $\omega(\xi)$: above this threshold, a frequency is considered "high".
- Assume $n \geq n_0$ then split

$$\left\| e^{tM^{-1}\partial_x} u_0 \right\|_{L^\infty} = \left\| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(\xi x - \omega(\xi)t)} \ \widehat{u_0}(\xi) \ \mathrm{d}\xi \right\|_{L^\infty_x}$$

$$\lesssim \sup_{c \in \mathbb{R}} \left| \int_{-n}^{n} e^{it(c\xi - \omega(\xi))} \ \widehat{u_0}(\xi) \ \mathrm{d}\xi \right| + \int_{|\xi|>n} |\widehat{u_0}(\xi)| \ \mathrm{d}\xi$$

$$= \text{Lo-Mid freq} \ + \ \text{Hi freq}$$

- Lo-Mid term can be bounded using Prop. 3.1 in the notes (itself a corollary of van der Corput, see Souganidis and Strauss '90):

$$\text{Lo-Mid} \lesssim \left(t^{-1/3} + t^{-1/2} n^{3/2} \right) \ \| u_0 \|_{L^1_x} .$$

Dispersive Est.: Proof Sketch 1

- One can determine a critical frequency magnitude $n_0 \approx 2$ based on stationary points of $\omega(\xi)$: above this threshold, a frequency is considered "high".
- Assume $n \geq n_0$ then split

$$\begin{aligned}\left\| e^{tM^{-1}\partial_x} u_0 \right\|_{L^\infty} &= \left\| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(\xi x - \omega(\xi) t)} \; \widehat{u_0}(\xi) \; \mathrm{d}\xi \right\|_{L^\infty_x} \\ &\lesssim \sup_{c \in \mathbb{R}} \left| \int_{-n}^{n} e^{it(c\xi - \omega(\xi))} \; \widehat{u_0}(\xi) \; \mathrm{d}\xi \right| + \int_{|\xi|>n} \left| \widehat{u_0}(\xi) \right| \; \mathrm{d}\xi \\ &= \text{Lo-Mid freq} \; + \; \text{Hi freq}\end{aligned}$$

- Lo-Mid term can be bounded using Prop. 3.1 in the notes (itself a corollary of van der Corput, see Souganidis and Strauss '90):

$$\text{Lo-Mid} \lesssim \left(t^{-1/3} + t^{-1/2} n^{3/2} \right) \; \|u_0\|_{L^1_x} .$$

Dispersive Est.: Proof Sketch 2

- For Hi term, bound is more pedestrian:

$$\begin{aligned}\int_{|\xi|>n} |\widehat{u_0}(\xi)| \, \mathrm{d}\xi &= \int_{|\xi|>n} \left(\langle\xi\rangle^s \, |\widehat{u_0}(\xi)|\right) \, \langle\xi\rangle^{-s} \, \mathrm{d}\xi \\ &\leq \left\|\langle\xi\rangle^s \widehat{u_0}(\xi)\right\|_{L^2_\xi} \left(\int_{|\xi|>n} \langle\xi\rangle^{-2s} \, \mathrm{d}\xi\right)^{1/2} \\ &\lesssim_s \|u_0\|_{H^s_x} \, n^{\frac{1}{2}-s}.\end{aligned}$$

- Putting Lo-Mid and Hi together, get

$$\|u(x,t)\|_{L^\infty} \lesssim \left(n^{\frac{1}{2}-s} + t^{-1/3} + t^{-1/2} n^{3/2}\right) \left(\|u_0\|_{L^1_x} + \|u_0\|_{H^s_x}\right).$$

- Choosing t suff. large and $n = t^{1/9}$ we see $s \geq 7/2$ needed to get Hi contribution dying faster than $t^{-1/3}$, DONE.

Dispersive Est.: Proof Sketch 2

- For Hi term, bound is more pedestrian:

$$\int_{|\xi|>n} |\widehat{u_0}(\xi)| \ \mathrm{d}\xi = \int_{|\xi|>n} \left(\langle\xi\rangle^s \ |\widehat{u_0}(\xi)|\right) \ \langle\xi\rangle^{-s} \ \mathrm{d}\xi$$
$$\leq \|\langle\xi\rangle^s \widehat{u_0}(\xi)\|_{L^2_\xi} \ \left(\int_{|\xi|>n} \langle\xi\rangle^{-2s} \ \mathrm{d}\xi\right)^{1/2}$$
$$\lesssim_s \|u_0\|_{H^s_x} \ n^{\frac{1}{2}-s}.$$

- Putting Lo-Mid and Hi together, get

$$\|u(x,t)\|_{L^\infty} \lesssim \left(n^{\frac{1}{2}-s} + t^{-1/3} + t^{-1/2} n^{3/2}\right) \left(\|u_0\|_{L^1_x} + \|u_0\|_{H^s_x}\right).$$

- Choosing t suff. large and $n = t^{1/9}$ we see $s \geq 7/2$ needed to get Hi contribution dying faster than $t^{-1/3}$, DONE.

Dispersive Est. for Nonlinear Case: Prelude

- Nice surprise: small nonlinear solns. also satisfy the same dispersive est.! This is basically all we need to prove scattering.
- Need the following easy bounds:

Lemma

Pick any $s > 0$. For any $p \in \mathbb{N}$ and any u in the appropriate fnc. space, we have

$$\left\| M^{-1}\partial_x u \right\|_{L^1} \lesssim \|u\|_{L^1},$$

$$\left\| M^{-1}\partial_x u \right\|_{H^s} \lesssim \|u\|_{H^s}, \quad \textit{and}$$

$$\left\| u^{p+1} \right\|_{H^s} \lesssim \|u\|_{L^\infty}^p \|u\|_{H^s}.$$

Dispersive Est. for Nonlinear Case: Prelude

- Nice surprise: small nonlinear solns. also satisfy the same dispersive est.! This is basically all we need to prove scattering.
- Need the following easy bounds:

Lemma

Pick any $s > 0$. For any $p \in \mathbb{N}$ and any u in the appropriate fnc. space, we have

$$\left\| M^{-1} \partial_x u \right\|_{L^1} \lesssim \|u\|_{L^1} ,$$

$$\left\| M^{-1} \partial_x u \right\|_{H^s} \lesssim \|u\|_{H^s} , \quad \textit{and}$$

$$\left\| u^{p+1} \right\|_{H^s} \lesssim \|u\|_{L^\infty}^{p} \|u\|_{H^s} .$$

Dispersive Est. for Nonlinear Case

Theorem (Nonlinear Solutions with Linear Decay, Albert '89, Dziubański-Karch '96)

Let $s \geq \frac{7}{2}$ *and suppose* $u_0 \in L^1(\mathbb{R}) \cap H^s(\mathbb{R})$*. If* $p > 4$ *then there exists* $\delta > 0$ *such that*

$$\|u_0\|_{L^1} + \|u_0\|_{H^s} < \delta$$

implies the soln. $u(x,t)$ *to GBBM satisfies*

$$\|u(x,t)\|_{L^\infty} \lesssim_{u_0} \langle t \rangle^{-1/3} \quad \forall\ t \in \mathbb{R}.$$

- Proof uses classical PDE tools: bootstrap estimates & Duhamel's formula to treat nonlinear term perturbatively.

Nonlinear Dispersive Est.: Proof Sketch 1

- Start by defining

$$q(t) \doteq \sup_{\tau\in[0,t]} \left[\|u(x,\tau)\|_{L_x^\infty} \langle\tau\rangle^{1/3} + \|u(x,\tau)\|_{H_x^s} \right].$$

  To prove claim (+ get extra persistence of H_x^s regularity!) it suffices to show $q(t)$ is bounded.

- I claim $q(t)$ is bounded if u_0 small and there is some $C > 0$ indep. of x, t, u_0 s.t.

$$q(t) \leq C \left(\|u_0\|_{L^1} + \|u_0\|_{H^s} + q(t)^{p+1} \right).$$

  Why? Assume above holds, then bootstrap boundedness of $q(t)$ (next two slides)

Nonlinear Dispersive Est.: Proof Sketch 2

Let $A > 1$ satisfy

$$\|\varphi\|_{L^\infty} \leq A \|\varphi\|_{H^s} \quad \forall\, \varphi \in H^s.$$

Pick $\eta \ll 1$ so that

$$\eta > C\,(3A\eta)^{p+1}\,.$$

Then, pick $\delta < \eta$ such that

$$\eta \geq C\left(\delta + (3A\eta)^{p+1}\right).$$

Having picked δ, we now suppose

$$\|u_0\|_{L^1} + \|u_0\|_{H^s} < \delta$$

as in the statement of the claim.

Nonlinear Dispersive Est.: Proof Sketch 3

By Sobolev embedding

$$q(0) \leq (1 + A)\delta < 2A\eta \quad \text{(conclusion holds at time 0).}$$

Additionally, if we **assume** $q(t) \leq 3A\eta$ for some particular t, then since

$$q(t) \leq C\left(\|u_0\|_{L^1} + \|u_0\|_{H^s} + q(t)^{p+1}\right).$$

we have

$$q(t) \leq C\left(\delta + (3A\eta)^{p+1}\right) \leq \eta < 2A\eta \quad \text{(assumption implies conclusion)}$$

Since $q(t)$ cts., the bootstrap principle then implies $q(t) \leq 2A\eta$ for all $t \geq 0$!

Thus it remains to prove

$$q(t) \leq C \left(\|u_0\|_{L^1} + \|u_0\|_{H^s} + q(t)^{p+1} \right).$$

Start using Duhamel form of GBBM (method of integrating factors): if $f(u) = (p+1)^{-1} u^{p+1}$,

$$u(x,t) = e^{tM^{-1}\partial_x} u_0 - \int_0^t e^{(t-\tau)M^{-1}\partial_x} M^{-1} \partial_x f(u(x,\tau)) \, \mathrm{d}\tau.$$

By the LBBM dispersive estimate and easy bounds from our lemma,

$$\langle t \rangle^{1/3} \|u(x,t)\|_{L^\infty} \lesssim \|u_0\|_{L^1} + \|u_0\|_{H^s} + \langle t \rangle^{\frac{1}{3}} \int_0^t \langle t-\tau \rangle^{-\frac{1}{3}} \left(\|u\|_{L^\infty}^{p-1} \|u\|_{H^s}^2 + \|u\|_{L^\infty}^{p} \|u\|_{H^s} \right) \, \mathrm{d}\tau$$

Nonlinear Dispersive Est.: Proof Sketch 4

Thus it remains to prove

$$q(t) \leq C \left(\|u_0\|_{L^1} + \|u_0\|_{H^s} + q(t)^{p+1} \right).$$

Start using Duhamel form of GBBM (method of integrating factors): if $f(u) = (p+1)^{-1} u^{p+1}$,

$$u(x,t) = e^{tM^{-1}\partial_x} u_0 - \int_0^t e^{(t-\tau)M^{-1}\partial_x} M^{-1} \partial_x f(u(x,\tau)) \, \mathrm{d}\tau.$$

By the LBBM dispersive estimate and easy bounds from our lemma,

$$\begin{aligned} \langle t \rangle^{1/3} \|u(x,t)\|_{L^\infty} &\lesssim \|u_0\|_{L^1} + \|u_0\|_{H^s} \\ &+ \langle t \rangle^{\frac{1}{3}} \int_0^t \langle t-\tau \rangle^{-\frac{1}{3}} \left(\|u\|_{L^\infty}^{p-1} \|u\|_{H^s}^2 + \|u\|_{L^\infty}^{p} \|u\|_{H^s} \right) \, \mathrm{d}\tau \end{aligned}$$

Nonlinear Dispersive Est.: Proof Sketch 5

Recognizing integrand as $\approx$ binomial expansion, can use $1 = \langle \tau \rangle \langle \tau \rangle^{-1}$ and definition of $q(t)$ to get

$$\langle t \rangle^{1/3} \|u(x,t)\|_{L^\infty} \lesssim \|u_0\|_{L^1} + \|u_0\|_{H^s} + q(t)^{p+1} \left(\langle t \rangle^{1/3} \int_0^t \langle t - \tau \rangle^{-1/3} \langle \tau \rangle^{(1-p)/3} \, \mathrm{d}\tau \right).$$

The integral is bounded unif. in t precisely when $p > 4$!

A similar arg. gives us a similar bound on $\|u\|_{H^s_x}$, hence we find

$$q(t) \leq C \left(\|u_0\|_{L^1} + \|u_0\|_{H^s} + q(t)^{p+1} \right).$$

as required.

Nonlinear Dispersive Est.: Proof Sketch 5

Recognizing integrand as $\approx$ binomial expansion, can use $1 = \langle\tau\rangle\langle\tau\rangle^{-1}$ and definition of $q(t)$ to get

$$\langle t\rangle^{1/3} \|u(x,t)\|_{L^\infty} \lesssim \|u_0\|_{L^1} + \|u_0\|_{H^s} + q(t)^{p+1}\left(\langle t\rangle^{1/3}\int_0^t \langle t-\tau\rangle^{-1/3}\langle\tau\rangle^{(1-p)/3}\,\mathrm{d}\tau\right).$$

The integral is bounded unif. in t precisely when $p > 4$!

A similar arg. gives us a similar bound on $\|u\|_{H^s_x}$, hence we find

$$q(t) \leq C\left(\|u_0\|_{L^1} + \|u_0\|_{H^s} + q(t)^{p+1}\right).$$

as required.

Re-statement of Main Theorem

Corollary (Scattering in H^1, Dziubański-Karch '96)

Under the same hypotheses as the previous theorem (small initial data, $p > 4$), there exist functions $u_{\pm}(x,t) \in C_t^1(\mathbb{R}; H_x^s)$ such that, if soln. $u(x,t)$ solves GBBM,

1. *$u_{\pm}$ both provide classical solutions to LBBM and*
2. *we have*

$$\|u_{\pm}(x,t) - u(x,t)\|_{H_x^1} \lesssim \langle t \rangle^{1-p/3}.$$

In particular,

$$\lim_{t \to \pm\infty} \|u_{\pm}(x,t) - u(x,t)\|_{H_x^1} = 0.$$

- Proof is basically just looking at consequences of nonlinear soln. satisfying linear dispersive est.

Re-statement of Main Theorem

Corollary (Scattering in H^1, Dziubański-Karch '96)

Under the same hypotheses as the previous theorem (small initial data, $p > 4$), there exist functions $u_\pm(x,t) \in C_t^1(\mathbb{R}; H_x^s)$ such that, if soln. $u(x,t)$ solves GBBM,

1. *$u_\pm$ both provide classical solutions to LBBM and*
2. *we have*

$$\|u_\pm(x,t) - u(x,t)\|_{H_x^1} \lesssim \langle t \rangle^{1-p/3}.$$

In particular,

$$\lim_{t \to \pm\infty} \|u_\pm(x,t) - u(x,t)\|_{H_x^1} = 0.$$

- Proof is basically just looking at consequences of nonlinear soln. satisfying linear dispersive est.

Proof of Main Thm.

Define

$$u_+(x,t) \doteq e^{tM^{-1}\partial_x}\left(u_0 - \int_0^\infty e^{-\tau M^{-1}\partial_x} M^{-1}\partial_x f(u(x,\tau))\,\mathrm{d}\tau\right),$$

which is obviously a solution to LBBM (one can show this is always finite, and in fact in $H^s_x\ \forall\ t$). u_- can be constructed by analogy.

Using Duhamel and product estimates, get

$$\|u_+ - u\|_{H^1_x} \lesssim \int_t^\infty \|u(x,\tau)\|^p_{L^\infty_x}\ \|u(x,\tau)\|_{H^1_x}\ \mathrm{d}\tau.$$

By conservation of energy and the dispersive est., this becomes

$$\|u_+ - u\|_{H^1_x} \lesssim_{u_0} \int_t^\infty \langle\tau\rangle^{-p/3}\,\mathrm{d}\tau \lesssim \langle t\rangle^{1-p/3} \quad \Rightarrow \quad \text{DONE!}$$

Proof of Main Thm.

Define

$$u_+(x,t) \doteq e^{tM^{-1}\partial_x}\left(u_0 - \int_0^\infty e^{-\tau M^{-1}\partial_x} M^{-1}\partial_x f(u(x,\tau))\, \mathrm{d}\tau\right),$$

which is obviously a solution to LBBM (one can show this is always finite, and in fact in $H_x^s\ \forall\ t$). u_- can be constructed by analogy.

Using Duhamel and product estimates, get

$$\|u_+ - u\|_{H_x^1} \lesssim \int_t^\infty \|u(x,\tau)\|_{L_x^\infty}^p\ \|u(x,\tau)\|_{H_x^1}\ \mathrm{d}\tau.$$

By conservation of energy and the dispersive est., this becomes

$$\|u_+ - u\|_{H_x^1} \lesssim_{u_0} \int_t^\infty \langle\tau\rangle^{-p/3}\, \mathrm{d}\tau \lesssim \langle t\rangle^{1-p/3} \quad \Rightarrow \quad \text{DONE!}$$

Discussion + Future Directions 1

- We know from Jan. 22 talk that $p = 1, 2$ are the major cases of physical interest... so how about scattering or non-scattering for these cases?
- By the same logic used to predict scattering for $p \gg 1$, we find that we can't expect nonlinearity to be a "small perturbation" for $p \approx 1$. This means very different methods must be used!
- One idea: look to tools for **generalized Korteweg-de Vries (GKdV)**, qualitatively similar to GBBM:

$$u_t + u_{xxx} + u^p u_x = 0.$$

- Hayashi & Naumkin '98 showed that small solns to GKdV scatter for $p > 2$.

Discussion + Future Directions 1

- We know from Jan. 22 talk that $p = 1, 2$ are the major cases of physical interest... so how about scattering or non-scattering for these cases?
- By the same logic used to predict scattering for $p \gg 1$, we find that we can't expect nonlinearity to be a "small perturbation" for $p \approx 1$. This means very different methods must be used!
- One idea: look to tools for **generalized Korteweg-de Vries (GKdV)**, qualitatively similar to GBBM:

$$u_t + u_{xxx} + u^p u_x = 0.$$

- Hayashi & Naumkin '98 showed that small solns to GKdV scatter for $p > 2$.

Discussion + Future Directions 2

- For $p = 1, 2$, GKdV is completely integrable (admits "true" solitons), so even for small Cauchy data we cannot expect scattering (and indeed we don't get it!)

Open Problem

Describe the long-time asymptotic behaviour of solutions to GBBM for $p \in [1, 4]$. In particular, is $p = 2$ the scattering threshold for GBBM as it is for GKdV?

- Issue: GBBM lacks many of the symmetries of GKdV, so certain methods (estimating vector fields) for GKdV can't be easily adapted to GBBM.
- My current work: take $p = 3$, study the modes that are "resonant" during the nonlinear self-interaction (see J.Kato & Pusateri 2011 for application of the method to cubic nonlinear Schrödinger eqn.).

Discussion + Future Directions 2

- For $p = 1, 2$, GKdV is completely integrable (admits "true" solitons), so even for small Cauchy data we cannot expect scattering (and indeed we don't get it!)

Open Problem

Describe the long-time asymptotic behaviour of solutions to GBBM for $p \in [1, 4]$. In particular, is $p = 2$ the scattering threshold for GBBM as it is for GKdV?

- Issue: GBBM lacks many of the symmetries of GKdV, so certain methods (estimating vector fields) for GKdV can't be easily adapted to GBBM.
- My current work: take $p = 3$, study the modes that are "resonant" during the nonlinear self-interaction (see J.Kato & Pusateri 2011 for application of the method to cubic nonlinear Schrödinger eqn.).

Discussion + Future Directions 2

- For $p = 1, 2$, GKdV is completely integrable (admits “true” solitons), so even for small Cauchy data we cannot expect scattering (and indeed we don’t get it!)

Open Problem

Describe the long-time asymptotic behaviour of solutions to GBBM for $p \in [1, 4]$. In particular, is $p = 2$ the scattering threshold for GBBM as it is for GKdV?

- Issue: GBBM lacks many of the symmetries of GKdV, so certain methods (estimating vector fields) for GKdV can’t be easily adapted to GBBM.
- My current work: take $p = 3$, study the modes that are “resonant” during the nonlinear self-interaction (see J.Kato & Pusateri 2011 for application of the method to cubic nonlinear Schrödinger eqn.).

Summary/Big Takeaways

- Basic intuition says solns. to GBBM with nonlinear term $u^p u_x$, $p \gg 1$, scatter to linear solns. Can prove this rigorously (with $p > 4$) via careful analysis of linear solns.
- The key tool underlying most of the hard analysis: dispersive estimate for LBBM. One can nearly guess the correct dispersive estimate using intuition from the method of stationary phase.
- Proving scattering is easy once one shows that initially small (in the right norm!) nonlinear solns. obey the same dispersive estimate as solns. to LBBM.
- To get scattering for lower p, one may need more powerful modern methods (ie. studying resonant interactions between normal modes in detail). Work in progress

Summary/Big Takeaways

- Basic intuition says solns. to GBBM with nonlinear term $u^p u_x$, $p \gg 1$, scatter to linear solns. Can prove this rigorously (with $p > 4$) via careful analysis of linear solns.
- The key tool underlying most of the hard analysis: dispersive estimate for LBBM. One can nearly guess the correct dispersive estimate using intuition from the method of stationary phase.
- Proving scattering is easy once one shows that initially small (in the right norm!) nonlinear solns. obey the same dispersive estimate as solns. to LBBM.
- To get scattering for lower p, one may need more powerful modern methods (ie. studying resonant interactions between normal modes in detail). Work in progress

Summary/Big Takeaways

- Basic intuition says solns. to GBBM with nonlinear term $u^p u_x$, $p \gg 1$, scatter to linear solns. Can prove this rigorously (with $p > 4$) via careful analysis of linear solns.
- The key tool underlying most of the hard analysis: dispersive estimate for LBBM. One can nearly guess the correct dispersive estimate using intuition from the method of stationary phase.
- Proving scattering is easy once one shows that initially small (in the right norm!) nonlinear solns. obey the same dispersive estimate as solns. to LBBM.
- To get scattering for lower p, one may need more powerful modern methods (ie. studying resonant interactions between normal modes in detail). Work in progress

Summary/Big Takeaways

- Basic intuition says solns. to GBBM with nonlinear term $u^p u_x$, $p \gg 1$, scatter to linear solns. Can prove this rigorously (with $p > 4$) via careful analysis of linear solns.
- The key tool underlying most of the hard analysis: dispersive estimate for LBBM. One can nearly guess the correct dispersive estimate using intuition from the method of stationary phase.
- Proving scattering is easy once one shows that initially small (in the right norm!) nonlinear solns. obey the same dispersive estimate as solns. to LBBM.
- To get scattering for lower p, one may need more powerful modern methods (ie. studying resonant interactions between normal modes in detail). Work in progress

Questions?