Isoperimetry and curvature: A detailed explanation

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0.1 Summary

We showed in class that we could get isoperimetry from functional inequalities (in $\mathbb{R}^n$), this report shows how to take this techniques further, to (certain) Riemannian manifolds.

0.2 Plan of the document

The idea is to define a language consisting of semigroups (def 1), a measure (def 4), an infinitesimal generator (def 6) and a second order "differential" operator (def 7) capable of encoding the geometry of an n-dimensional Riemannian Manifold in terms of applications of this objects to (Lipschitz) functions. In this way, one can replicate the ideas of functional inequalities (section 4), that we have in the Euclidean setting and preserve the techniques and (to certain extent) some proofs that we used in class to tackle the isoperimetric problem in a coordinate-free manner.
It is clear that not every manifold can satisfy iso-perimetric inequalities, so we look for conditions to guarantee it. The condition $\mathrm{CD}(k, n)$ (Def 16) bounds the Carre-du-Champ operator (def 7) from below by k times the norm of the gradient of the function and $1/n$ the infinitesimal generator squared. This condition, referred to as the Bakry-Emery curvature-dimension condition ensures some kind of isoperimetry. That is what we show in this report.

- Sections [1-3] develop the general framework of Markov Triples and the BE curvature-dimension condition.
- Section [4] describes functional inequalities
- Section [5] presents the proof of Ledoux of linear isoperimetry under the Bakry-Emery curvature-dimension condition

Definition 1. *(Semigroup of operators)*
A family of operators $\{P_t\}_{t\geq 0}$ defined on some set V of real functions on (X, S) is called a semigroup if it satisfies:

- *For every t, P_t is a linear operator, sending bounded measurable functions to bounded measurable functions.*
- *$P_0 = I$ (Initial condition)*
- *$P_{t+s} = P_t \circ P_s$ (Semigroup property)*

Definition 2. *(Positivity preserving)*
A semigroup as above is said to be positivity preserving if

$$f \geq 0 \textit{ implies } P_t f \geq 0 \textit{ for all } t.$$

Definition 3. *(Conservative)*
A semigroup as above is said to be conservative if $\mathbf{1} \in V$

$$P_t \mathbf{1} = \mathbf{1} \textit{ for all } t.$$

where $\mathbf{1}$ denotes $\mathbf{1}(x) = 1 \; \forall x$

Definition 4. *(Invariant measure)*
A measure μ, is said to be invariant with respect to a semigroup of linear operators $\{P_t\}_{t\geq 0}$ if

$$\int P_t f d\mu = \int f d\mu$$

for all bounded measurable f.

In the following we denote: $B_b = \{f : X \to \mathbb{R} | f \text{ is measurable and bounded}\}$

Definition 5. *(Markov semigroup)*
A positivity-preserving, conservative semigroup on B_b with invariant measure μ is said to be a Markov semigroup if it is strongly continuous on $L^2(X, \mu)$

Definition 6. *(Infinitesimal Generator and it's domain)*
For a semigroup $\{P_t\}_{t\geq 0}$ over $L^2(X, \mu)$, we define the infinitesimal generator as the pair $(L, \mathcal{D}_L)$, where

$$\mathcal{D}_L = \left\{ f \in L^2(X, .\mu) : \lim_{t \downarrow 0} \frac{Ptf - f}{t} \textit{ exists} \right\} \textit{ and } L(f) = \lim_{t \downarrow 0} \frac{Ptf - f}{t} \textit{ for } f \in \mathcal{D}_L$$

Let $\mathcal{A}$ be an algebra of $\mathcal{D}_L$ functions.

Definition 7. *(Carré du Champ operator)*
We define the Carré du Champ operator, $\Gamma : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$, of a Markov semigroup $\{P_t\}$ as follows:

$$\Gamma(f, g) = \frac{1}{2}(L(fg) - fLg - gLf)$$

It is obvious from the definition that the operator is bilinear and symmetric.
Often we are interested in $\Gamma(f, f)$ which is usually denoted by $\Gamma(f)$.

Observation 1. *(Domain of Γ)*
Note that at this point the definition of Γ depends on the algebra $\mathcal{A}$, we need to specify the largest domain in which we can define the Carré Du Champ. We will require a lot of properties from this set so that the operator is useful.

Definition 8. *(Extended algebra)*
Let $\mathcal{A}$ be an algebra, we say that $\mathcal{A}^{ext}$ is an extended algebra of $\mathcal{A}$ if it satisfies

1. $f \in \mathcal{A}^{ext}, h \in \mathcal{A} \Rightarrow hf \in \mathcal{A}$
2. $f \in \mathcal{A}^{ext},\ \forall h \in \int fhd\mu \geq 0 \Rightarrow f \geq 0$
3. $\forall f \in \mathcal{A}^{ext}\ \forall \varphi \in \mathcal{C}^2, \varphi(0) = 0,\ \varphi \circ f \in \mathcal{A}^{ext}$
4. *L (and Γ) can be extended from $\mathcal{A}$ to $\mathcal{A}^{ext}$*
5. $\Gamma(f) \geq 0\ \forall f \in \mathcal{A}^{ext}$
6. *The diffusion property holds for (L, Γ) in $\mathcal{A}^{ext}$*
7. *If $f \in \mathcal{A}^{ext}$ and $g \in \mathcal{A}$,* $\int \Gamma(f, g)d\mu = -\int fLgd\mu = -\int gLfd\mu$
8. $f \in \mathcal{A} \Rightarrow P_t f \in \mathcal{A}^{ext}$

Definition 9. *(Associated Dirichlet form)*
In the same setting, whenever we have defined Γ (the Carré du Champ) we define it's associated Dirichlet form to be the function $\mathcal{E} : \mathcal{A} \to \mathbb{R}$

$$\mathcal{E}(f) = \int \Gamma(f)d\mu$$

The Dirichlet form can be defined in a set larger than $\mathcal{A}$, this set is named the domain of the Dirichlet form

Definition 10. *(Domain of a Dirichlet form)*

$$\mathcal{D}(\mathcal{E}) = \left\{ f \in L^2(X,\mu) : \lim_{t\downarrow 0} \frac{1}{t} \int f(f - P_t f) d\mu \text{ exists} \right\}$$

Definition 11. *(Markov triple)*
Let $\{P_t\}$ be a Markov semigroup, with invariant measure μ and let Γ be it's associated Carré du Champ operator, then (X, μ, Γ) is called a Markov triple.

Definition 12. *(Ergodic)*
A Markov triple (X, Γ, μ) is said to be ergodic if $\forall f \in \mathcal{D}(L)$

$$Lf = 0 \Rightarrow f \text{ is constant}$$

Let $\Gamma_0(f, g) = fg$ (the product of the functions), then the Carré du Champ Γ_1 is the operator:

$$\Gamma_1(f, g) = L(\Gamma_0(f, g)) - \Gamma_0(f, Lg) - \Gamma_0(Lf, g)$$

This rewriting teaches us how to generate more operators in the same space, by repeating this procedure, if we restrict our domain to be L-stable, meaning that if $f \in \hat{\mathcal{A}}$ then $Lf \in \hat{\mathcal{A}}$. For an integer k we write:

$$\Gamma_{k+1} = L\Gamma_k(f, g) - \Gamma_k(f, Lg) - \Gamma_k(Lf, g)$$

1 Markov Triples on compact Riemannian Manifolds

Our setting will be a smooth connected Riemannian manifold M.

For $W \in \mathcal{C}^\infty(M)$, with finite integral, assume without loss of generality that

$$\int_M e^{-W} dV = 1$$

(change W by a constant if not).
We can use e^{-W} as a Radon-Nikodyn derivative and get a probability measure μ that is absolutely continuous with respect to the standard volume, i.e.

$$\mu(A) = \int_M \mathbf{1}_A e^{-W} dV$$

If M is a compact Riemannian manifold we write $\mathcal{A} = \mathcal{C}^\infty(M)$ but if M is assumed to be a Riemannian manifold (not nec. compact) we put $\mathcal{A} = \mathcal{C}_c^\infty(M)$, $\mathcal{A}^{ext} = \mathcal{C}^\infty(M)$ and we can write

$$\Gamma(f, g) = \langle \nabla f, \nabla g \rangle$$

.

Definition 13. *(Density of a Riemannian manifold)*
In the setting as above we say that M has a smooth density e^{-W}

Corollary 1. *In our setting, of compact Riemannian manifolds,*

$$\Gamma_2(f) = |\nabla\nabla f|^2 + \mathrm{Ricc}_g(\nabla f, \nabla f)$$

2 Assumptions on a Markov Triple

Definition 14. *(Standard Markov Triple)*
A Markov triple (X, Γ, μ) is said to be standard if it is

1. *A diffusion markov triple.*
2. *Ergodic*
3. *Conservative*

Until now, the theory developed has been very general. The final framework will be the full markov triple:

Definition 15. *(Full Markov Triple)*
A full Markov Triple is a Standard Markov triple in which we additionally require:

- Γ *(and also L) are defined on an extended algebra.*
- *If $f \in \mathcal{A}$ is such that $\Gamma(f) = 0$ then f must be constant.*
- *L (defined originally in $\mathcal{A}$ not in $\mathcal{A}^{ext}$) has a unique self-adjoint extension.*

From now on, when we mention Markov Triples we will be referring to Full Markov Triples.

3 Bakry-Emery curvature dimension condition

Definition 16. *(CD(k, n))*
We say that the curvature dimension inequality is satisfied for the pair (k, n) (or simpler CD(k, n)) if

$$\Gamma_2(f) \geq k\Gamma_1(f) + \frac{1}{n}(Lf)^2$$

The name curvature dimension inequality comes from obtaining lower bounds for Γ_2 using Bochner-Weitzenböck identity, and relating the first part to a curvature bound relating the Ricci curvature and the dimension coming from a simple inequality in terms of the HS norm of the Hessian. For details the reference is again [1] pg 70-72.

Definition 17. *CD*(k,∞)
As before, we say that $CD(k,\infty)$ *is satisfied if*

$$\Gamma_2(f) \geq k\Gamma_1(f)$$

4 Functional inequalities in the Markov setting

Let us write Var_μ for the variance of the measure μ, that is,

$$Var_\mu(f) = \int f^2 d\mu - \left(\int f d\mu\right)^2$$

Definition 18. *(Global Poincaré inequality for Markov Triples)*
We say that a Markov triple (X, Γ, μ) *satisfies the Poincaré inequality with constant* k *if for every function* $f \in \mathcal{A}$

$$Var_\mu(f) \leq k\mathcal{E}(f)$$

Definition 19. *(Local Poincaré inequality for Markov Triples)*
We say that a Markov triple satisfies the local Poincaré inequality with constant k *if for every function* $f \in \mathcal{A}$

$$P_t f^2 - (P_t f)^2 \leq \frac{1 - e^{-2kt}}{k} P_t(\Gamma f) \tag{1}$$

Definition 20. *(Local reverse Poincaré inequality for Markov Triples)*
We say that a Markov triple satisfies the reverse Poincaré inequality with constant k *if for every function* $f \in \mathcal{A}$

$$P_t f^2 - (P_t f)^2 \geq \frac{e^{2kt} - 1}{k} \Gamma(P_t f)$$

Definition 21. *(Second version of the Poincaré inequality for Markov Triples)*
For a Markov triple (X, Γ, μ) *we say it satisfies Poincaré and reverse Poincaré inequalities if*

$$2t\Gamma(P_t f) \leq P_t(f^2) - (P_t f)^2 \leq 2tP_t(\Gamma(f))$$

Observation 2. *Note that directly from the reverse Poincaré inequality, one finds Lipschitz properties of the Carré du Champ operator,* Γ*, with respect to the semigroup: If* $0 \leq f \leq 1$ *then by the reverse Poincaré inequality (second version)*

$$2t\Gamma(P_t(f)) \leq P_t(f^2) - (P_t f)^2 \leq P_t(f^2) \leq P_t(f) \leq P_t(\mathbf{1})$$

And so, as the semigroup is conservative, we have $2tP_t(f) \leq 1$*which of course can be rewritten in a way that it evokes Lipschitz properties:*

$$\Gamma(P_t f) \leq \frac{1}{2t}$$

Definition 22. *(Isoperimetric profile) In a measure metric space* (X, d, μ) *the isoperimetric profile* $I_\mu : [0, \infty) \to \mathbb{R}$ *is defined by*

$$I_\mu(x) = \inf\{\mu^+(A) : \mu(A) = x\}$$

where the infimum is taken over all Borel sets A*.*

Definition 23. *(Isoperimetric-type inequality and linear isoperimetry)*
We say that an isoperimetric type inequality is satisfied if we can find a function $i : [0, \infty) \to \mathbb{R}$ *that lower bounds* I_μ*, namely*

$$I_\mu(x) \geq i(x) \ \ \forall x \in [0, \infty)$$

We say that linear isoperimetry holds if $i(x) = c \min\{x, 1 - x\}$ *for some constant* $c > 0$*.*

Example 1. *(Guassian isoperimetry) Let* μ *be the Gaussian measure in* $\mathbb{R}$ *and let* φ *be it's density and* Ψ *it's accumulative distribution, that is*

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \Psi(x) = \int_{-\infty}^{x} \varphi(s) ds$$

Then,

$$I_\mu(x) \geq \varphi\left(\Psi^{-1}(x)\right)$$

For the Lebesgue measure, the isoperimetric problem has been solved and we can address it in two ways:

- Via Brunn-Minkowski's inequality (**How we did it in class)**
- Using Optimal Transport.

Let us show that the n-ball is the (Lebesgue measure) isoperimetric minimizer in $\mathbb{R}^n$ using optimal transport:

Let A be any measurable set and let B be the n-dimensional ball, so let us transport A onto B.
Let μ be the uniform measure on A and ν the uniform measure on B then we know by Optimal transport:

$$\frac{1}{|A|} = \det(\nabla\phi)\frac{1}{|B|}$$

where ϕ is a convex function transporting μ onto ν. Then, by integrating over A

$$|A|^{1-\frac{1}{n}} \leq \frac{1}{n|B|^{\frac{1}{n}}}\int_{\partial A}\phi\cdot\mathbf{n}dx \leq \frac{1}{n|B|^{\frac{1}{n}}}|\partial A|$$

Now set $|A| = |B|$ to find a minimizer among all sets that have this measure, and get

$$n|A| \leq |\partial A|$$

which is satisfied with equality by the ball.

In different settings, people avoid using μ^+: A set E is said to have finite perimeter if $\mathbf{1}_E$ is of bounded variation. Then the perimeter is defined:

$$\mathcal{P}(E) = ||D\mathbf{1}_E||$$

This is a key idea will use approximation of "derivatives of indicator functions".

Theorem 1. *(Linear isoperimetry implies Poincaré)*
In the setting of a Markov Triple (X, Γ, μ) if linear isoperimetry holds with constant c then the Poincaré inequality holds for constant $\frac{4}{c^2}$

In $\mathbb{R}^n$ to prove this kind of results we make use of the famous Co-area formula, in this setting we need a similar idea:

Lemma 1. *(Co-area inequality for the Minkowski boundary measure)*
In the setting of a Markov Triple (X, Γ, μ) for every 1-Lipschitz function f we have

$$\int_{-\infty}^{\infty}\mu^+(\{x\in X : f(x) > r\})dr \leq \int\sqrt{\Gamma(f)}d\mu$$

Now further, having functional inequalities translate on having concentration of measure:

Observation 3. *Having a Poincaré inequality is equivalent to having exponential decay in variance. Having a log-sobolev inequality is equivalent to having exponential decay in entropy, this both statements imply conditions on the concentration of the measure.*

Theorem 2. *(Poincaré and exponential integrability)*
If the Poincaré inequality is satisfied with constant c, then for every 1-Lipschitz function f and $s < \sqrt{\frac{4}{c}}$

$$\int e^{sf} d\mu < \infty$$

Observation 4. *Using* $s = \frac{1}{\sqrt{c}}$ *together with the Markov exponential inequality one gets:*

$$\mu\left(f \geq \int f d\mu + r\right) \leq 3e^{-r/c||f||_{Lip}}$$

Theorem 3. *(Log-sobolev and exponential square integrability)*
If the log-sobolev inequality is satisfied with constant c then for every 1-Lipschitz function f and every $\sigma^2 < \frac{1}{c}$ *one has*

$$\int e^{f^2/2\sigma^2} d\mu < \infty$$

Observation 5. *In a similar fashion as 4 we get*

$$\mu\left(\left|f - \int f d\mu\right| \geq r\right) \leq 2e^{-r^2/(2c)||f||_{Lip}}$$

Observation 6. *The standard gaussian satisfies the log-sobolev inequality with constant 1.*

Now we know that if we have isoperimetry this implies some functional inequalities. We also know that this functional inequalities imply concentration inequalities.
Q: Do you expect concentration to imply isoperimetry?
Q: Under what conditions do you think we can deduce isoperimetry from concentration?

5 Semigroup proof for linear Isoperimetry

Definition 24. *(Exponential concentration)*
We say that (X, Γ, μ) *satisfies exponential concentration with constants* c, C *if for every integrable* $1-$*Lipschitz function* f *one has*

$$\mu\left(f \geq \int f d\mu + r\right) \leq Ce^{-cr}$$

Theorem 4. *(Milman) [1] [Theorem 8.7.1] Under* $\Gamma_2 \geq 0$, *if exponential concentration holds with constants* c *and* C, *then*

$$I_\mu(x) \geq c' \min\{x, 1-x\}, \;\; x \in [0, 1]$$

Theorem 5. *(Milman) [2] [Theorem 1] Under* $\Gamma_2 \geq 0$, *if* $\alpha_\mu(r) \to 0$ *as* $r \to \infty$ *then*

$$I_\mu(x) \geq c_2^{'} \min\{x, 1-x\}, \ \ x \in [0,1]$$

Observation 7. *As we will see in the proof, the constant* c' *does not depend on the dimension of the underlying manifold.*

Lemma 2. *(Poincaré and log-Sobolev inequalities imply Lipschitz properties)*
Let $0 \leq f \leq 1 \in \mathcal{A}$ *then*

- *If the local Poincaré inequality is satisfied then*

$$\Gamma(P_t f) \leq \frac{1}{2t}$$

- *If the local log-Sobolev inequality is satisfied then*

$$\Gamma(P_t f) \leq \frac{1}{t}(P_t f)^2 \log \frac{1}{P_t f} \leq \frac{1}{t} \log \frac{1}{P_t f}$$

Observation 8. *In this case it is helpful to study a larger class of functions, namely* $A_0^{const} = A + \mathbb{R}$ *where* $+$ *denotes the Minkowski sum.*

The proof presented here is a (way) more detailed version of the one found in [1].

Proof. Let $f \in A_0^{const}$ then

$$\int f \log f d\mu - \int P_t f \log P_t f d\mu = -\int_0^t \frac{d}{ds}\left(\int P_s f \log P_s f d\mu\right) ds$$

Now we can use De Bruijn's identity and get

$$\int f \log f d\mu - \int P_t f \log P_t f d\mu = \int_0^t \int \frac{\Gamma(P_s f)}{P_s f} d\mu \tag{2}$$

For Theorem 5 we need

$$\int f^2 d\mu - \int (P_t f)^2 d\mu = 2\int_0^t \int \Gamma(P_s f) d\mu ds \tag{\textbf{3.1*}}$$

Observation 9. *It is clear even from this point that the details for 5 are a little easier.*

Now as $\text{CD}(0,\infty)$ is part of our assumptions, local log-Sobolev inequalities hold. So we can use local Poincaré and the reverse local Poincaré inequalities. To use them, suppose further that $0 < \epsilon \leq f \leq 1$ then by the reverse Poincaré inequality

$$\frac{\Gamma(P_s f)}{P_s f} \leq \frac{1}{s}\left[P_s(f \log f) - P_s f \log P_s f\right]$$

But $1 \geq f \leq \epsilon$ means that $f \log f \geq 0$ and by positivity preserving property of P_s we have $P_s(f \log f) < 0$. We can use this in the last inequality to get a bound:

$$\frac{\Gamma(P_s f)}{P_s f} \leq \frac{1}{s}\left[-P_s f \log P_s f\right] = \frac{1}{s}\left[P_s f \log \frac{1}{P_s f}\right] \leq \frac{1}{s}[P_s f \log\left(\frac{1}{\epsilon}\right)]$$

And by simply taking $P_s f$ to the other side we get a bound for our Carré du champ evaluated in the semigroup:

$$\Gamma(P_s f) \leq \frac{1}{s} \log\left(\frac{1}{\epsilon}\right)(P_s f)^2$$

which by considering square roots can be rewritten as follows

$$\sqrt{\Gamma(P_s f)} \leq \sqrt{\frac{1}{s}\log\left(\frac{1}{\epsilon}\right)} P_s f$$

.

We can now use this bound,

$$\int \frac{\Gamma(P_s f)}{P_s f} d\mu = \int \frac{\sqrt{\Gamma(P_s f)}\sqrt{\Gamma(P_s f)}}{P_s f} d\mu \leq \sqrt{\frac{1}{s}\log\left(\frac{1}{\epsilon}\right)} \int \sqrt{\Gamma(P_s f)} d\mu \tag{3}$$

But under our assumption CD$(0, \infty)$ the strong gradient bounds hold, so we have $\sqrt{\Gamma(P_s f)} \leq P_s(\sqrt{\Gamma(f)})$, plugging the bound in 3 gives

$$\int \frac{\Gamma(P_s f)}{P_s f} d\mu \leq \sqrt{\frac{1}{s}\log\left(\frac{1}{\epsilon}\right)} \int \sqrt{\Gamma(f)} d\mu$$

To return to bounding in 2 we need to integrate with respect to s from 0 to t so we have

$$\int f \log f d\mu - \int P_t f \log P_t f d\mu \leq \int_0^t \sqrt{\frac{1}{s}\log\left(\frac{1}{\epsilon}\right)} \int \sqrt{\Gamma(f)} d\mu ds \tag{4}$$

For 5 we need the easier estimate:

$$\int f^2 d\mu - \int (P_t f)^2 d\mu \leq 2\sqrt{2t} \int \sqrt{\Gamma(f)} d\mu \tag{\textbf{3.4*}}$$

As only the first term depends on s on the bound, we can explicitly calculate the integral $\int_0^t \sqrt{(1/s)} ds = 2t^{1/2}$ so we get

$$\int f \log f d\mu - \int P_t f \log P_t f d\mu \leq 2\sqrt{t \log\left(\frac{1}{\epsilon}\right)} \int \sqrt{\Gamma(f)} d\mu \tag{5}$$

Observation 10. *In order to use the exponential concentration property, we need to use Lipschitz functions, therefore we can obtain the following lemma.*

Lemma 3. *Under* $\Gamma_2 \geq 0$, $-\psi = -\sqrt{\log \frac{2}{P_t f}}$ *is* $\frac{1}{2\sqrt{t}}$ *- Lipschitz with respect to* Γ.

Proof. Note that we know

$$\Gamma\left(\sqrt{\log \frac{1}{P_t f}}\right) = \frac{1}{4} \frac{1}{\log \frac{1}{P_t f}} \frac{1}{(P_t f)^2} \Gamma(P_t f)$$

By the Log-sobolev inequality, we can bound

$$\Gamma\left(\sqrt{\log \frac{1}{P_t f}}\right) \leq -\frac{1}{4} \frac{1}{\log \frac{1}{P_t f}} \frac{1}{P_t f} \frac{2}{2t} P_t f \log P_t f$$

which by getting the sign inside the logarithm and cancelling the remaining terms is equivalent to

$$\Gamma\left(\sqrt{\log \frac{1}{P_t f}}\right) \leq \frac{1}{4t}$$

Now that we have a Lipschitz function, we can apply the hypothesis of exponential concentration, namely:

$$\mu\left(-\psi \geq \int -\psi d\mu + r\right) = \mu\left(\psi \leq \int \psi d\mu - r\right) \leq Ce^{-2cr\sqrt{t}}$$

Now we aim to use properties of ψ, namely the following lemma

Lemma 4. *The map* $u \to \sqrt{\log \frac{2}{u}}$ *is convex for* $u \in (0, 1]$.

Proof. of the lemma: By differentiablility it is enough to show that the second derivative is non-negative in $(0, 1]$.

$$\frac{d}{du}\sqrt{\log \frac{2}{u}} = \frac{2\log \frac{2}{x} - 1}{4x^2(\log \frac{2}{x})^{3/2}}$$

So the map is convex if $2\log \frac{2}{x} - 1 \geq 0$ which happens if $x < \frac{2}{\sqrt{e}}$. As $\frac{2}{\sqrt{e}} > 1$, we get that $u \to \sqrt{\log \frac{2}{u}}$ is convex for $u \in (0, 1]$.

By Jensen's inequality, as we know by the lemma that the function is convex,

$$\sqrt{\log \frac{2}{\int f d\mu}} = \sqrt{\log \frac{2}{\int P_t f d\mu}} \leq \int \psi d\mu$$

And so we can bound,

$$\mu\left(\psi \leq \sqrt{\log \frac{2}{\int f d\mu}} - r\right) \leq \mu\left(\psi \leq \int \psi d\mu - r\right) \leq Ce^{-2cr\sqrt{t}}$$

Claim 1. *Let* $0 \leq r \leq \sqrt{\frac{1}{2}\log \frac{2}{\int f d\mu}}$ *If it holds that*

$$\mu\left(\psi \leq \sqrt{\log \frac{2}{\int f d\mu}} - r\right) \leq Ce^{-2cr\sqrt{t}}$$

then

$$\mu\left(P_t f \geq \sqrt{2\int f d\mu e^{r^2}}\right) \leq Ce^{-2cr\sqrt{t}}$$

Proof of claim. As we do not use specific properties of $P_t f$ and $\int f d\mu$, and the prove is only to work with the inequalities, we can write for easier understanding $y := P_t f$ and $x = \int f d\mu$, so our hypothesis are

$$\sqrt{\log \frac{2}{y}} \leq \sqrt{\log \frac{2}{x}} - r \text{ and also } 0 \leq r \leq \sqrt{\frac{1}{2}\log \frac{2}{x}} \tag{6}$$

and our aim is to conclude that

$$y \geq \sqrt{2x} e^{r^2} \tag{7}$$

Observe that

$$0 \leq (r - \frac{1}{2}\sqrt{\log \frac{2}{x}})^2 = r^2 - r\log \frac{2}{x} + \frac{1}{4}\log \frac{2}{x}$$

which by adding and substracting $\frac{1}{2}\log \frac{2}{x}$ yields

$$\frac{1}{2}\log \frac{2}{x} - r^2 \leq \left(\sqrt{\log \frac{2}{x}} - r\right)^2 \tag{8}$$

As our hypothesis ensures the lower bound is positive so we can consider square roots on both sides and get

$$\sqrt{\frac{1}{2}\log \frac{2}{x} - r^2} \leq \sqrt{\log \frac{2}{x}} - r$$

To prove the claim, as we have sets for which the inequalities hold, it is enough to show that the hypothesis is necessary for the result, meaning that in order for 7 to hold we must have that 6 holds. So 7 holds if and only if

$$\log \frac{2}{y} \leq \frac{1}{2} \log \frac{2}{x} - r^2$$

so by using 8 we have

$$\sqrt{\log \frac{2}{y}} \leq \sqrt{\log \frac{2}{x}} - r$$

This means that whenever 7 holds 6 also holds. So by returning to our original problem we have

$$\mu\left(P_t f \geq \sqrt{2\int f d\mu e^{r^2}}\right) \leq \mu\left(\psi \leq \sqrt{\log \frac{2}{\int f d\mu}} - r\right) \leq Ce^{-2cr\sqrt{t}} \tag{9}$$

For 5 we need not to bound with the exponential of course, but with the concentration function so we have

$$\mu\left(P_t f \geq \sqrt{2\int f d\mu e^{r^2/4t}}\right) \leq \alpha_\mu(r) \tag{\textbf{3.8*}}$$

Returning to our original problem, suppose further that $0 < \delta \leq 1$ is such that $\delta \geq \sqrt{2\int f d\mu e^{r^2}}$ then we can use it to bound one of the terms in the Poincaré inequality, namely,

$$\int P_t f \log \frac{1}{P_t f} d\mu \geq \log \frac{1}{\delta} \int_{\{P_t \leq \delta\}} P_t f d\mu = \left(\int f d\mu - \int_{\{P_t f > \delta\}} P_t f d\mu\right) \log \frac{1}{\delta}$$

where the first inequality comes from considering the integral over a subset and the equality by writing $\{P_t f > \delta\} = X \setminus \{P_t f \leq \delta\}$ and using invariance.
As $\epsilon \leq f \leq \mathbf{1}$ by conservativeness $\epsilon \leq P_t f \leq \mathbf{1}$, meaning that $-P_t f \geq -\mathbf{1}$. So we can bound the last term as follows

$$\int P_t f \log \frac{1}{P_t f} d\mu \geq \left(\int f d\mu - \int_{\{P_t f > \delta\}} \mathbf{1} d\mu\right) \log \frac{1}{\delta} \geq \left(\int f d\mu - Ce^{-2cr\sqrt{t}}\right) \log \frac{1}{\delta} \tag{10}$$

where of course the last inequality comes from applying 9 □

5.0.1 KEY APPROXIMATION

Here we arrive to the core of the proof, until now even though a lot of inequalities and claims have been proved we haven't seen how this connects to isoperimetry. This step, fundamental to our analysis, shows the true relation between functional inequalities and isoperimetry. As we will see, if the functional inequality involves derivatives and we are able to approximate indicator functions, we will be able to understand isoperimetric type inequalities in such space.

5.0.2 Important assumption

In this section we adopt an additional assumption to the framework of Markov triples (X, Γ, μ) then for every $f \in \mathcal{D}(\mathcal{E})$

$$\sqrt{\Gamma(f)} = \limsup_{d(x,y)\to 0} \frac{|f(x) - f(y)|}{d(x,y)}$$

Notice that this is the case of Riemannian manifolds. This assumptions is restrictive but it is good enough for our purposes because most (if not all) of our examples fulfill it. Note that $\sqrt{\Gamma(f)}$ is the *pointwise Lipschitz constant* of f at every x with respect to the absolute value.
For our theory, it is fundamental to understand the expression $\Gamma(\mathbf{1}_A)$ where A is some closed set. Nevertheless as this function may not be in $\mathcal{A}$ or in $\mathcal{D}(\mathcal{E})$ we need to approximate it. Let us attempt such approximation by putting

$$f_\epsilon(x) = \max\left\{1 - \frac{1}{\epsilon} d(x, A), 0\right\}$$

So let us analyze such limits, meaning we aim to compute

$$\sqrt{\Gamma(f_\epsilon)} = \limsup_{d(x,y)\to 0} \frac{|f_\epsilon(x) - f_\epsilon(y)|}{d(x,y)}$$

Just to have everything written explicitly, this is the same as

$$\sqrt{\Gamma(f_\epsilon)} = \limsup_{d(x,y)\to 0} \frac{\left|\max\left\{1 - \frac{1}{\epsilon} d(x, A), 0\right\} - \max\left\{1 - \frac{1}{\epsilon} d(y, A), 0\right\}\right|}{d(x,y)} \tag{11}$$

We solve this by analizing cases of x, meaning that we see the sets to which x belongs and draw conclusions of the limit superior in each set.

Claim 2.

$$\sqrt{\Gamma(f_\epsilon)} \leq \left(\frac{1}{\epsilon}\right) \mathbf{1}_{A_\epsilon \setminus A}$$

Proof of claim. The relevant cases are determined by the right hand side of the claim so we can reduce our proof to $x \in A_\epsilon \setminus A$ and it's complement.

- If $x \in A$, as the limit superior is the infimum of the tail suprema, and the inner most term has an absolute value, we now that 0 is a lower bound for the suprema. But if $x \in A$ and y_n is a sequence such that $y_n \in A$ and $d(y_n, x) \to 0$ then we have $d(x, A) = 0 = d(y_n, A)$ so the numerator on 11 is 0. So the lower bound is attained and by the infimum property, $\sqrt{\Gamma(f_\epsilon)} = 0$.

- If $x \notin A_\epsilon$ then $\max\left\{1 - \frac{1}{\epsilon}d(x, A), 0\right\} = 0$ and so the numerator in 11 is only depending on y. So (by considering subsequences without loss of generality) we have two options, $y_n \in A_\epsilon$ or $y \notin A_\epsilon$, but the limit superior is the infimum of sequential limits, and as the latter produce 0 in the numerator, the lower bound 0 is achieved and this $\Gamma(f_\epsilon) = 0$.

- If $x \in A_\epsilon \setminus A$ then by using a sequence $y_n \in A_\epsilon \setminus A$, equation 11 becomes

$$\limsup_{d(x,y)\to 0} \frac{|\frac{1}{\epsilon}(d(x, A) - d(y, A))|}{d(x, y)}$$

But for every $z \in A$ we have the trinagle inequality

$$d(x, z) \leq d(x, y) + d(y, z)$$

so considering the infimum over all $z \in A$ this becomes

$$d(x, A) \leq d(x, y) + d(y, A)$$

which by reversing the roles of x and y gives

$$|d(x, A) - d(y, A)| \leq d(x, y)$$

Using this we get

$$\sqrt{\Gamma(f_\epsilon)} \leq \limsup_{d(x,y)\to 0} \frac{1}{\epsilon}\frac{d(x, y)}{d(x, y)} = \frac{1}{\epsilon}$$

And the claim is proved.

But $\mathbf{1}_A$ may not be in the domain of Γ so we want to approximate what the value of $\sqrt{\Gamma(\mathbf{1}_A)}$ would be, recall that $f_\epsilon \to \mathbf{1}_A$ pointwise so let us write $1_A = \liminf_{\epsilon\to 0} f_\epsilon$.

$$\int \sqrt{\liminf_{\epsilon\to 0} \Gamma(f_\epsilon)}d\mu = \int \limsup_{d(x,y)\to 0} \liminf_{\epsilon\to 0} \frac{|f_\epsilon(x) - f_\epsilon(y)|}{d(x, y)}d\mu$$

But as the superior limit is an infimum and the inferior limit is a supremum we can bound this by the reversed limit and apply the claim, namely

$$\int \liminf_{\epsilon\to 0} \sqrt{\Gamma(f_\epsilon)}d\mu \leq \int \liminf_{\epsilon\to 0} \limsup_{d(x,y)\to 0} \frac{|f_\epsilon(x)-f_\epsilon(y)|}{d(x,y)}d\mu \leq \int \liminf_{\epsilon\to 0} \frac{1}{\epsilon}\mathbf{1}_{A_\epsilon\setminus A}d\mu$$

Now naturally in the last term we use Fatou's lemma and get

$$\int \liminf_{\epsilon\to 0} \sqrt{\Gamma(f_\epsilon)}d\mu \leq \liminf_{\epsilon\to 0} \frac{1}{\epsilon}\int \mathbf{1}_{A_\epsilon\setminus A}d\mu = \liminf_{\epsilon\to 0} \frac{1}{\epsilon}(\mu(A_\epsilon)-\mu(A)) = \mu^+(A)$$

□

Observation 11. *This bound is fundamental to our analysis. It is the connection between isoperimetric type inequalities and Markov semigroups.*

Now we are almost finished, we need to organize all the results and do a suitable parameter choice.

Summary 1. *We first proved, which says:*

$$\int f\log f d\mu + \int P_t f \log\frac{1}{P_t f}d\mu \leq 2\sqrt{t\log\left(\frac{1}{\epsilon}\right)}\int\sqrt{\Gamma(f)}d\mu$$

Then we showed that if δ and r satisfy some conditions, we have 10 for $f\in\mathcal{A}$

$$\left(\int f d\mu - Ce^{-2cr\sqrt{t}}\right)\log\frac{1}{\delta} \leq \int P_t f\log\frac{1}{P_t f}d\mu$$

and we then showed

$$\int\sqrt{\Gamma(\mathbf{1}_A)}d\mu \leq \mu^+(A)$$

Now notice that $\epsilon \leq f \leq 1$ directly implies that

$$-\epsilon\log\frac{1}{\epsilon} \leq \int f\log f d\mu$$

Now we can put all of this bounds together, meaning that both terms in the left of 5 are bounded from below and the last term on the right is bounded from above so we get

$$-\epsilon\log\frac{1}{\epsilon} + \left(\mu(A) - Ce^{-2cr\sqrt{t}}\right)\log\frac{1}{\delta} \leq 2\sqrt{t\log\left(\frac{1}{\epsilon}\right)}\mu^+(A) \tag{12}$$

Furthermore, as $0<\epsilon<1$ we know $(1-\epsilon)\mu(A)\leq\mu(A)$ so we have

$$-\epsilon\log\frac{1}{\epsilon} + \left((1-\epsilon)\mu(A) - Ce^{-2cr\sqrt{t}}\right)\log\frac{1}{\delta} \leq 2\sqrt{t\log\left(\frac{1}{\epsilon}\right)}\mu^+(A) \tag{13}$$

5.0.3 Parameter choice

Now we can choose δ, r, ϵ as long as they satisfy the conditions in the lemmata.
Let us pick

- $\epsilon = \mu(A)^2$
- $r^2 = \frac{1}{4}\log\frac{1}{\mu(A)} = \frac{1}{8}\log\frac{1}{\epsilon}$
- $0 < \sqrt{2}\mu(A)^{1/4} \le \delta \le 2\mu(A)^{1/4}$.
  But to ensure that δ is smaller than 1 we need $2\mu(A)^{1/4} \le 1$ i.e. $\mu(A) \le 16$.

Let us first verify that this choice is valid in the sense that δ, r, ϵ meet their conditions.
For ϵ we just need $0 < \epsilon \le 1$ which holds trivially as $\mu(X) = 1$.
Our condition for r is that $0 \le r \le \sqrt{\frac{1}{2}\log\frac{2}{\int f d\mu}} = \sqrt{\frac{1}{2}\log\frac{2}{\mu(A)}}$ but $\frac{1}{4}\log\frac{1}{\mu(A)} \le \frac{1}{2}\log\frac{2}{\mu(A)}$ so this one also holds.
For δ we required $\delta \ge \sqrt{2\int f d\mu e^{r^2}} = \sqrt{2\mu(A)}e^{(1/4)\log\frac{1}{\mu(A)}} = \sqrt{2\mu(A)}\frac{1}{\mu(A)^{1/4}} = \sqrt{2}\mu(A)^{1/4}$
And so every condition is satisfied and we can plug in this values in 13 and we get

$$-2\mu(A)^2\log\frac{1}{\mu(A)} + \log\frac{1}{\delta}\left[(1-\mu(A)^2)\mu(A) - Ce^{-2cr\sqrt{t}}\right] \le 4\sqrt{2}r\sqrt{t}\mu(A)^+$$

And $\delta \le 2\mu(A)^{1/4}$ implies $\log\frac{1}{\delta} \ge \frac{1}{4}\log\frac{1}{16\mu(A)}$ so we can further lower bound our term and get

$$-2\mu(A)^2\log\frac{1}{\mu(A)} + \frac{1}{4}\log\frac{1}{16\mu(A)}\left[(1-\mu(A)^2)\mu(A) - Ce^{-2cr\sqrt{t}}\right] \le 4\sqrt{2}r\sqrt{t}\mu(A)^+$$

But now as $\mu(A)^2 < \frac{1}{2}$ we have

$$-2\mu(A)^2\log\frac{1}{\mu(A)} + \frac{1}{4}\log\frac{1}{16\mu(A)}\left[\frac{\mu(A)}{2} - Ce^{-2cr\sqrt{t}}\right] \le 4\sqrt{2}r\sqrt{t}\mu(A)^+ \tag{14}$$

Similarly, for 5 we have

$$\mu(A) - \left(2\mu(A) + \frac{r}{\sqrt{2t}}\right)^2 - \alpha_\mu(r) \tag{\textbf{3.14*}}$$

And finally we choose t such that $r\sqrt{t} = \frac{1}{2c}\log\frac{4C}{\mu(A)}$ and obtain

$$-2\mu(A)^2 \log \frac{1}{\mu(A)} + \frac{1}{16} \log \frac{1}{16\mu(A)} \mu(A) \leq 2\sqrt{2} \log \left(\frac{4C}{\mu(A)} \right) \mu^+(A)$$

From which we obtain that there exists c' such that if $0 < \mu(A) < c'$ we have

$$c'\mu(A) \leq \mu^+(A)$$

.

So we have proved that if $x = \mu(A)$ we have

$$c'x \leq I_\mu(x)$$

Proposition 1. *(At most linear growth of the isoperimetric profile)*
For the isoperimetric profile I_μ, if $\mathrm{Ricc}_\psi \geq 0$ we have

$$\frac{I_\mu(x)}{x} \text{ is a non-increasing function of } x$$

Now if $\frac{I_\mu(x)}{x}$ is non-increasing, suppose that $x > 1 - x$ then

$$c' \leq \frac{I_\mu(x)}{x} \leq \frac{I_\mu(x)}{1-x}$$

So finally we arrive to

$$c' \min\{x, 1-x\} \leq I_\mu(x)$$

This concludes the proof of theorem 4 as for the case $1 - x > x$ we just reverse the roles. We remark that almost the same proof works (by changing the exponential for α and picking suitable t) to prove theorem 5.

References

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