Long-time Asymptotics for the Cubic Nonlinear Schrödinger Equation

Approach via the Method of Space-Time Resonances

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Overview

1. Intro to the PDE, loose statement of main result
2. Facts about linear Schrödinger eqn
3. Heuristic computation underlying asymptotic analysis
4. Highlights of rigorous proof of main result

- Today, we study the Cauchy (initial value) problem for the **cubic nonlinear Schrödinger equation (NLS)**:

$$\begin{cases} iu_t + \frac{1}{2}\Delta u + \lambda |u|^2 u = 0 & \forall\ (t,x) \in \mathbb{R} \times \mathbb{R}^d \\ u|_{t=0}(x) = u_0(x) & \forall\ x \in \mathbb{R}^d \end{cases}$$

$u(t,x)\colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$, $u_0\colon \mathbb{R}^d \to \mathbb{C}$, and $\lambda = \pm 1$

Introduction to the PDE

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- NLS is a universal model for the evolution of wavepackets of the form

$$U(t,x) = u(t,x)e^{i(\xi_0 x - \omega_0 t)}$$

  in nonlinear dispersive systems (w/ $\xi_0, \omega_0 \in \mathbb{R}$ s.t. $u(x,t)$ changes "slowly" compared to the sinusoidal term)

Main Questions

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Suppose $d = 1$. How do solns of NLS behave as $t \to \infty$? If u_0 is sufficiently small, does the nonlinearity eventually become negligible?

- Main question is a baby step towards understanding stability of special solutions (solitons)
- Technically, we need to have a soln that's a bit better than L_x^2 to say something about the main question

The Best Answer to the Main Question

Theorem (Main Theorem, Loose Version)

Suppose

$$\|u_0(x)\|_{H^1_x} + \|xu_0(x)\|_{L^2_x} \leq \epsilon \ll 1.$$

Then, the unique global-in-time solution to NLS satisfies the following asymptotics: there exists some small $\beta > 0$ and two (unique) bounded, real-valued functions $F(\xi), \phi(\xi)$ such that

$$u(t,x) = (it)^{-1/2} F(x/t) \exp\left(\frac{i|x|^2}{2t} + i\,|F(x/t)|^2 \log t + i\phi(x/t)\right) + \mathcal{O}\left(t^{-1/2-\beta}\right).$$

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- The weighted norm does not scale nicely wrt NLS scaling symmetry, so *cannot remove smallness hypothesis*
- We shall also see that the $\log t$ frequency correction represents *deviation from linear behaviour*
- Small technical detail: we'll contruct $F(\xi)$ as *a priori* complex-valued, but WLOG we can absorb its argument into $\phi(\xi)$

Today's Approach to the Main Theorem

- We follow J. Kato & Pusateri 2011: use **method of space-time resonances (STR)** to establish a framework for getting the right asymptotics
- STR starts from a formal computation based on the stationary phase lemma... proof then reduces to justifying the steps of the computation (usually by bootstrapping)

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- STR introduced by Germain, Masmoudi, & Shatah in 2009, has been applied to other problems including the water waves equations
- Method does not require "structural assumptions" on PDE $\Rightarrow$ hope for STR working with difficult PDEs like the Benjamin-Bona-Mahony eqn!

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- Today, we will use part of their approach, but we do not control weighted norms with **method of vector fields** as they do

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- Today, we will use part of their approach, but we do not control weighted norms with **method of vector fields** as they do
- Deift & Zhou 2003: used **inverse scattering theory**
- Lindblad & Soffer 2006: used a clever ansatz based on soln of linear Schrödinger, then constructed an iterative procedure
- Ifrim & Tataru 2015: used the **method of testing by wave packets**

Linear Schrödinger Review 1

- The Cauchy problem for the linear Schrödinger equation reads

$$\begin{cases} iu_t + \frac{1}{2}\partial_x^2 u = 0 \quad \forall\ (t,x) \in \mathbb{R} \times \mathbb{R} \\ u|_{t=0}(x) = u_0(x) \quad \forall\ x \in \mathbb{R} \end{cases}$$

- Soln can be written in terms of the Fourier transform $u_0(x) \mapsto \widehat{u_0}(\xi)$:

$$u(t,x) = e^{\frac{it}{2}\partial_x^2} u_0 \doteq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\left(\xi x - \frac{t}{2}\xi^2\right)}\ \widehat{u_0}(\xi)\ \mathrm{d}\xi.$$

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- Immediately, we find by Plancherel that

$$\left\| e^{\frac{it}{2}\partial_x^2} u_0 \right\|_{L_x^2} = \|u_0\|_{L_x^2}$$

Linear Schrödinger Review 2

- After some contour integration, can represent Schrödinger flow as a convolution in spatial variable:

$$e^{\frac{it}{2}\partial_x^2} u_0 = (2\pi i t)^{-1/2} \int_{-\infty}^{+\infty} \mathrm{d}y \ \exp\left(\frac{i|x-y|^2}{2t}\right) \ u_0(y).$$

- Can simplify if taking $t \gg 1$:

Lemma (Linear Schrödinger Asymptotics)

We can write

$$e^{\frac{it}{2}\partial_x^2} u_0 = (it)^{-1/2} \, e^{\frac{ix^2}{2t}} \, \widehat{u_0}\left(\frac{x}{t}\right) + r(t,x)$$

and there exists $\kappa > 0$ *so that*

$$\|r(t,x)\|_{L_x^\infty} \lesssim t^{-1/2-\kappa} \, \|x u_0(x)\|_{L_x^2}$$

From spatial convolution representation of soln,

$$e^{\frac{it}{2}\partial_x^2}u_0 = (2\pi it)^{-1/2}\int_{-\infty}^{+\infty}\mathrm{d}y\ \exp\left(\frac{ix^2}{2t}-\frac{ixy}{t}+\frac{iy^2}{2t}\right)\ u_0(y)$$

$$= (it)^{-1/2}\exp\left(\frac{ix^2}{2t}\right)\widehat{u_0}\left(\frac{x}{t}\right)$$

$$+\left[(2\pi it)^{-1/2}\exp\left(\frac{ix^2}{2t}\right)\int_{-\infty}^{+\infty}\mathrm{d}y\ u_0(y)\ e^{-ixy/t}\left(e^{iy^2/2t}-1\right)\right]$$

where we added and subtracted $e^{-ixy/t}$ to the integrand. Define $r(t,x)$ to be stuff in square brackets above.

Proof of the Lemma 2

It's an easy exercise to show, for any $\kappa < \frac{1}{2}$, have

$$\left| e^{iy^2/2t} - 1 \right| \lesssim_\kappa t^{-\kappa} \langle y \rangle^{2\kappa}$$

where $\langle y \rangle^2 = 1 + y^2$.

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Thus since $\langle y \rangle^{-\zeta} \in L^1_x$ for $\zeta > 1/2$, Cauchy-Schwarz gives

$$\begin{aligned} \| r(t,x) \|_{L^\infty_x} &\lesssim t^{-\frac{1}{2}-\kappa} \int_{-\infty}^{+\infty} \mathrm{d}y \; \langle y \rangle^{2\kappa} \, |u_0(y)| \\ &\lesssim t^{-\frac{1}{2}-\kappa} \left\| \langle x \rangle^{2\kappa+\zeta} u_0(x) \right\|_{L^2_x} . \end{aligned}$$

To finish the proof, shrink κ until $2\kappa + \zeta \leq 1$.

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NOTE: This proof is where we find weighted norm is unavoidable!

An Instructive Heuristic 1

- If $u(t,x)$ satisfies NLS, define its **profile** by

$$f(t,x) = e^{-\frac{it}{2}\partial_x^2} u(t,x)$$

- Notice: if $u(t,x)$ was a linear wave, then $f = f(x)$ hence $\partial_t f \neq 0$ implies genuinely nonlinear behaviour

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- Notice: if $u(t,x)$ was a linear wave, then $f = f(x)$ hence $\partial_t f \neq 0$ implies genuinely nonlinear behaviour
- We'll write out NLS in terms of the profile *in Fourier space*:

$$\partial_t \hat{f} = i\lambda \; e^{\frac{it}{2}\xi^2} \left(|u|^2 u\right)^{\wedge}.$$

Need to write RHS all in terms of $\hat{f}$... use convolution theorem

- After some calculation (see notes), get the following: define a phase function by

$$\Phi(\eta, \sigma; \xi) \doteq \eta(\xi - \sigma),$$

then the transformed profile $\hat{f}(t, \xi)$ obeys the ODE

$$\partial_t \hat{f} = \frac{i\lambda}{2\pi} \int \mathrm{d}\eta \int \mathrm{d}\sigma \; e^{it\Phi(\eta,\sigma;\xi)} \; \hat{f}\,(t, \xi - \eta) \;\; \hat{f}(t, \sigma) \; \overline{\hat{f}}(t, \sigma - \eta).$$

- RHS is like an oscillatory integral (pretend $\hat{f}$ varies slowly in time compared to exponential term), so can use classical tools to estimate for $t \gg 1$

What are Resonances?

Definition

We call a pair (η_0, σ_0) *a* space resonance *if*

$$\nabla_{\eta,\sigma}\Phi(\eta_0, \sigma_0) = 0$$

or a time resonance *if*

$$\Phi\left(\eta_0, \sigma_0\right) = 0.$$

If (η_0, σ_0) *is both a space resonance and a time resonance, we call it a* space-time resonance*.*

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- **Method of stationary phase** $\Rightarrow$ dominant contributions to RHS of profile ODE for $t \gg 1$ come from space-time resonances

Asymptotic Approx. of Profile ODE

- After computing space-time resonances and plugging into the stationary phase lemma, find for $t \gg 1$ that

$$\partial_t \hat{f}(t,\xi) \approx i\lambda t^{-1} \left|\hat{f}(t,\xi)\right|^2 \hat{f}(t,\xi)$$

- Above implies that $\partial_t \left|\hat{f}(t,\xi)\right|^2 \approx 0$ for $t \gg 1$ so $\exists\ F(\xi)$ s.t.

$$\left|\hat{f}(t,\xi)\right|^2 \approx |F(\xi)|^2$$

- Thus we can integrate the approximate ODE to get (up to a ξ-dependent phase correction)

$$\hat{f}(t,\xi) \approx F(\xi)\ e^{i\lambda |F(\xi)|^2 \log t}$$

No Convergence to Linear Dynamics!

- Thus we find that for $t \gg 1$

$$\hat{u}(t,\xi) \approx F(\xi)\exp\left[-\frac{it}{2}\xi^2 + i\lambda|F(\xi)|^2\log t\right]$$

- By rough analogy with linear Schrödinger asymptotics we then expect

$$u(t,x) \approx (it)^{-1/2}\,F(x/t)\,\exp\left(\frac{i|x|^2}{2t} + i\,|F(x/t)|^2\log t\right)$$

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- This differs from linear behaviour by logarithmic frequency correction!

Roadmap to the Main Theorem

1. Get local existence in the right norm, trivial
2. Need to show small solns to NLS at least decay in time like linear solns:
$$\|u(t,x)\|_{L_x^\infty} \lesssim \langle t\rangle^{-1/2}$$
plus some other weighted norm bounds. Requires a long bootstrap argument! Main contribution of Kato & Pusateri 2011.
3. From here, follow Hayashi & Naumkin 1998: rewrite the ODE for the profile using an "integrating factor", get a sequence converging to asymptotic profile amplitude $F(\xi)$.
4. Concluding the main asymptotic expansion is very easy from here!

Some Norms

$\mathcal{S}'(\mathbb{R})$ = tempered distributions on $\mathbb{R}$

Definition (Weighted Sobolev Spaces)

$$H_x^{s,\ell} = \left\{ u \in \mathcal{S}'(\mathbb{R}) \;\middle|\; \|u\|_{H_x^{s,\ell}} \doteq \left\| \langle x \rangle^{\ell} \langle \partial_x \rangle^{s} u \right\|_{L_x^2} < \infty \right\}$$

Similarly, we have

$$\dot{H}_x^{s,\ell} = \left\{ u \in \mathcal{S}'(\mathbb{R}) \;\middle|\; \|u\|_{\dot{H}_x^{s,\ell}} \doteq \left\| \langle x \rangle^{\ell} \left| \partial_x \right|^{s} u \right\|_{L_x^2} < \infty \right\}$$

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Definition (Bootstrap Norm)

In terms of a given $T > 0$ *and some small* $\alpha \ll 1$ *to be determined later,*

$$\|u\|_{X_T} = \left\| t^{1/2} u \right\|_{L_t^\infty L_x^\infty} + \left\| t^{-\alpha} u \right\|_{L_t^\infty \dot{H}_x^{1,0}} + \left\| t^{-\alpha} f \right\|_{L_t^\infty H_x^{0,1}} + \|u\|_{L_t^\infty L_x^2}$$

Local-in-time Soln

Proposition (Local Well-Posedness)

Given $\epsilon > 0$ sufficiently small and a function $u_1(x)$ satisfying

$$\|u_1\|_{H_x^{1,0} \cap H_x^{0,1}} \leq \epsilon,$$

there exists $T > 1$ and a unique solution

$$u \in C\left([0, T]; H_x^{1,0}(\mathbb{R}) \cap H_x^{0,1}(\mathbb{R})\right)$$

to NLS satisfying

$$\|u\|_{X_T} \lesssim \epsilon.$$

- We have switched to prescribing Cauchy data at $t = 1$ (always OK w/ "real" Cauchy problem by shrinking ϵ more if necessary)

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- To prove linear decay of $u(t,x)$ enough to control $\|u\|_{X_T}$
- To do this, we use the bootstrap principle and first prove that

$$\|u\|_{X_T} \leq \epsilon + C \|u\|_{X_T}^3 \quad (\star)$$

where ϵ controls size of Cauchy data

Sketch of the Bootstrap Argument 2

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- Rigorously: start with an X_T-small local solution constructed earlier, flow out again for a short time, then use $\star$ to show we always stay X_T-small
- Iterate to prove global existence with linear decay
- So: it remains to prove $\star$...

Proof of $\star$: L_x^2-type bounds, part 1

Lemma

Let $u(t,x)$ be the local solution constructed to NLS constructed earlier. There exists some constant $C > 0$ such that

$$\sup_{t\in[1,T]} t^{-\alpha}\left[\|u(t,x)\|_{\dot{H}_x^{1,0}} + \|f(t,x)\|_{H_x^{0,1}}\right] \leq \epsilon + C\,\|u\|_{X_T}^3 .$$

- The only hard part involves the bound on $\|f(t,x)\|_{H_x^{0,1}}$, we sketch the proof on next slide

Proof of $\star$: L^2_x-type bounds, part 2

- Recall that the profile ODE is

$$\partial_t \hat{f} = \frac{i\lambda}{2\pi} \int \mathrm{d}\eta \int \mathrm{d}\sigma \; e^{it\eta(\xi-\sigma)} \; \hat{f}\,(t, \xi-\eta) \;\; \hat{f}(t,\sigma) \; \overline{\hat{f}}(t,\sigma-\eta)$$

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  Use this to get all our bounds
- To estimate $\|xf(t,x)\|_{L^2_x}$, obvs. enough to estimate $\left\|\partial_\xi \hat{f}(t,x)\right\|_{L^2_\xi}$ using ODE above (integrate wrt time)
- When ∂_ξ hits the exponential term $e^{it\eta(\xi-\sigma)}$, gain an extra factor of t! Bad news

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- When ∂_ξ hits the exponential term $e^{it\eta(\xi-\sigma)}$, gain an extra factor of t! Bad news
- Avoid this issue by changing variables $\sigma \mapsto \xi - \sigma$ so phase turns into $\eta\sigma$. Only doable bcz of special structure of nonlinearity

Proof of $\star$: L^∞_x-type bounds, part 1

- We start by strategically isolating the leading-order behaviour in the profile ODE:

Lemma

We may decompose $\partial_t \hat{f}(t,\xi)$ into the following form:

$$\partial_t \hat{f}(t,\xi) = i\lambda t^{-1} |\hat{f}(t,\xi)|^2 \hat{f}(t,\xi) + R(t,\xi)$$

and there exists $\delta \in \left(3\alpha, \frac{1}{4}\right)$ such that

$$|R(t,\xi)| \lesssim t^{-1-\delta+3\alpha} \|u\|^3_{X_T}.$$

- Skip the proof today, mostly computation and easy bounds

Proof of $\star$: L_x^∞-type bounds, part 2

Lemma

$$\sup_{t\in[1,T]} t^{1/2}\left\|u\right\|_{L_x^\infty} \leq \epsilon + C\left\|u\right\|_{X_T}^3$$

Proof of $\star$: L_x^∞-type bounds, part 2

Lemma

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- To prove this, start with previous result:

$$\partial_t \hat{f}(t,\xi) = i\lambda t^{-1}|\hat{f}(t,\xi)|^2\hat{f}(t,\xi) + R(t,\xi)$$

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- Pretend $\left|\hat{f}(t,\xi)\right|^2$ does not depend on $\hat{f}$, then we can simplify the profile ODE by introducing an *integrating factor*:

$$B(t,\xi) \doteq \lambda \int_1^t \frac{\mathrm{d}s}{s}\,\left|\hat{f}(s,\xi)\right|^2$$

$$\Rightarrow \partial_t\left(\hat{f}(t,\xi)e^{-iB(t,\xi)}\right) = R(t,\xi)e^{-iB(t,\xi)}$$

Proof of ⋆: L^∞_x-type bounds, part 3

- Integrating wrt time gives

$$\widehat{w}(t,\xi) \doteq \hat{f} e^{-iB} = \hat{f}(1,\xi) + \int_1^t \mathrm{d}s\ R(s,\xi)\ e^{-iB(s,\xi)}$$

Proof of $\star$: L^∞_x-type bounds, part 3

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- Can replace $\hat{f}$ with $\widehat{w}$ any time we need a bound in terms of $|\hat{f}|$: good since $\widehat{w}$ is controlled easily in terms of ICs and remainder R!

Proof of $\star$: L^∞_x-type bounds, part 3

- Integrating wrt time gives

$$\widehat{w}(t,\xi) \doteq \hat{f} e^{-iB} = \hat{f}(1,\xi) + \int_1^t \mathrm{d}s \; R(s,\xi)\; e^{-iB(s,\xi)}$$

- Can replace $\hat{f}$ with $\widehat{w}$ any time we need a bound in terms of $|\hat{f}|$: good since $\widehat{w}$ is controlled easily in terms of ICs and remainder R!
- By linear Schrödinger asymptotics lemma applied to $u = e^{\frac{it}{2}\partial_x^2} f$ there is some $\kappa \ll 1$ such that

$$\|u(t,x)\|_{L^\infty_x} \leq t^{-1/2} \left\|\hat{f}(t,\xi)\right\|_{L^\infty_\xi} + t^{-1/2-\kappa} \|f(t,x)\|_{H^{0,1}_x}.$$

- First term dealt with by remainder estimate, second one by earlier L^2_x−type estimates.

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Existence of Asymptotic Profile

- We know

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- Since

$$B(t,\xi) = \lambda \int_1^t \frac{\mathrm{d}s}{s} \; \left| \hat{f}(s,\xi) \right|^2 \approx \lambda |F(\xi)|^2 \log t$$

we should therefore try to prove

$$\lim_{t\to\infty} \widehat{w}(t,\xi) = F(\xi).$$

ie. show $\widehat{w}(t,\xi)$ is Cauchy wrt t in L^∞_ξ. *Easy using control of $R(t,\xi)$!*

- Now, we want to conclude

$$u(t,x) = (it)^{-1/2} F(x/t) \exp\left(\frac{i|x|^2}{2t} + i\,|F(x/t)|^2 \log t + i\phi(x/t)\right) + \mathcal{O}\left(t^{-1/2-\beta}\right)$$

- Existence of $\phi(\xi)$ follows from same tricks used to get $F(\xi)$, as well as

$$B(t,\xi) \to \lambda |F(\xi)|^2 \log t + \phi(\xi)$$

- Combine above with $\widehat{w}(t,\xi) \to F(\xi)$ and

$$\|u\|_{X_T} \lesssim \epsilon$$

to conclude (after doing some accounting in the error term)

Final Remarks

- Very similar arguments work to get asymptotics for Hartree equation (nonlinear term $= \left(|x|^{-1} * |u^2|\right) u$) or *systems* of NLS with nonlinear term $= |u_a|^2 u_b$

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- Space-time resonance strategy may need to be massaged for some difficult systems like mKdV (combine with method of vector fields)
- Optimistic about a space-time resonance approach to generalized Benjamin-Bona-Mahony (more on this in the future)!

Questions?