1 Differentiation

Recall that the derivative (or first derivative) of $y = f(x)$ is given by

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}.$$

Some other common notation for the derivative of $y = f(x)$ include

$$f'(x), \quad \frac{d}{dx} f(x), \quad \frac{dy}{dx}, \quad \frac{df}{dx}.$$

The second derivative is the derivative of the first derivative,

$$\frac{d^2}{dx^2} f(x) = \frac{d}{dx}\Big(\frac{d}{dx} f(x)\Big).$$

Some other common notation for the second derivative of $y = f(x)$ include

$$f''(x), \quad \frac{d^2}{dx^2} f(x), \quad \frac{d^2 y}{dx^2}, \quad \frac{d^2 f}{dx^2}.$$

1.1 Table of Derivatives

Elementary Functions

$$\frac{d}{dx} x^a = a x^{a-1}, \quad a \in \mathbb{R} \tag{1}$$

$$\frac{d}{dx} |x| = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} \tag{2}$$

Exponential and Logarithms

$$\frac{d}{dx} e^x = e^x \tag{3}$$

$$\frac{d}{dx} \ln(x) = \frac{1}{x} \tag{4}$$

$$\frac{d}{dx} a^x = \ln(a) \cdot a^x \tag{5}$$

$$\frac{d}{dx} \log_a(x) = \frac{1}{x \ln(a)} \tag{6}$$

Trigonometric Functions

$$\frac{d}{dx} \sin(x) = \cos(x) \tag{7}$$

$$\frac{d}{dx} \cos(x) = -\sin(x) \tag{8}$$

$$\frac{d}{dx} \tan(x) = \sec^2(x) \tag{9}$$

$$\frac{d}{dx} \cot(x) = -\csc^2(x) \tag{10}$$

$$\frac{d}{dx} \sec(x) = \tan(x) \sec(x) \tag{11}$$

$$\frac{d}{dx} \csc(x) = -\cot(x) \csc(x) \tag{12}$$

$$\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1 - x^2}} \tag{13}$$

$$\frac{d}{dx} \cos^{-1}(x) = -\frac{1}{\sqrt{1 - x^2}} \tag{14}$$

$$\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1 + x^2} \tag{15}$$

$$\frac{d}{dx} \cot^{-1}(x) = -\frac{1}{1 + x^2} \tag{16}$$

$$\frac{d}{dx} \sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2 - 1}} \tag{17}$$

$$\frac{d}{dx} \csc^{-1}(x) = -\frac{1}{|x|\sqrt{x^2 - 1}} \tag{18}$$

Hyperbolic Functions

$$\frac{d}{dx} \sinh(x) = \cosh(x) \tag{19}$$

$$\frac{d}{dx} \cosh(x) = \sinh(x) \tag{20}$$

$$\frac{d}{dx} \tanh(x) = \operatorname{sech}^2(x) \tag{21}$$

1.2 Differentiation Rules

We now state several rules that will allow us to differentiate combinations of the basic functions

Linearity: If $a, b \in \mathbb{R}$,

$$\frac{d}{dx}(af(x) + bg(x)) = af'(x) + bg'(x).$$

Product Rule:

$$\frac{d}{dx}\Big(f(x) \cdot g(x)\Big) = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$

Quotient Rule:

$$\frac{d}{dx}\Big(\frac{f(x)}{g(x)}\Big) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$

Chain Rule:

$$\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x).$$

Derivative of Inverse:

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}.$$

1.3 Tangent and Normal Line Approximations

Using the derivative, we can construct the tangent lines and normal lines to the curve $y = f(x)$. If $f'(a)$ exists, then

1. Tangent Line: The tangent line to the curve $y = f(x)$ at the point $(a, f(a))$ is given by

$$y = f(a) + f'(a)(x - a).$$

2. Normal Line: The normal line to the curve $y = f(x)$ at the point $(a, f(a))$ is given by

$$y = f(a) - \frac{1}{f'(a)}(x - a).$$

1.4 Rates of Change

The derivative tells us the instantaneous rate of change of a function with respect to one of its inputs.

1.4.1 Motion of a Particle:

Let $s(t)$ denote the *position* of the particle at time t. We have the following definitions:

1. The *velocity* of the particle at time t is $v(t) = s'(t)$.
2. The *speed* of the particle at time t is $|v(t)|$.
3. The *acceleration* of the particle at time t is $a(t) = v'(t) = s''(t)$.

1.5 Example Problems

1.5.1 Using the Limit Definition to Derive Limits

Problem 1.1. ($\star$) Using the definition of the derivative, derive the derivative of $f(x) = x^7$.

Solution 1.1. We use the alternate definition of the derivative,

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}.$$

In this case, we have

$$\begin{aligned} f'(x) &= \lim_{y \to x} \frac{y^7 - x^7}{y - x} \\ &= \lim_{y \to x} \frac{(y - x)(y^6 + y^5x + y^4x^2 + y^3x^3 + y^2x^4 + yx^5 + x^6)}{y - x} && \text{Difference of Powers} \\ &= \lim_{y \to x} (y^6 + y^5x + y^4x^2 + y^3x^3 + y^2x^4 + yx^5 + x^6) \\ &= 7x^6. \end{aligned}$$

Problem 1.2. ($\star$) Using the definition of the derivative, derive the derivative of $f(x) = \sqrt{x}$.

Solution 1.2. From the definition of the derivative, we have

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{x + h} - x}{h} \cdot \frac{\sqrt{x + h} + \sqrt{x}}{\sqrt{x + h} + \sqrt{x}} \\ &= \lim_{h \to 0} \frac{x + h - x}{h(\sqrt{x + h} + \sqrt{x})} \\ &= \lim_{h \to 0} \frac{1}{\sqrt{x + h} + \sqrt{x}} \\ &= \frac{1}{2\sqrt{x}}. \end{aligned}$$

Problem 1.3. ($\star$) Using the definition of the derivative, derive the derivative of $f(x) = \ln(x)$ for $x > 0$.

Solution 1.3. From the definition of the derivative, we have

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\ln(x + h) - \ln(x)}{h} \\ &= \lim_{h \to 0} \frac{1}{h} \ln\left(\frac{x + h}{x}\right) \\ &= \ln \lim_{h \to 0} \left(1 + \frac{1}{x} \cdot h\right)^{\frac{1}{h}} && \text{Continuity} \\ &= \ln(e^{\frac{1}{x}}) && \lim_{h \to 0} \left(1 + ah\right)^{\frac{1}{h}} = \lim_{n \to \pm\infty} \left(1 + \frac{a}{n}\right)^{\frac{1}{n}} = e^a \\ &= \frac{1}{x}. \end{aligned}$$

Problem 1.4. ($\star$) Using the definition of the derivative, derive the derivative of $f(x) = \sin(x)$.

Solution 1.4. From the definition of the derivative, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\ &= \sin(x) \cdot \lim_{h\to 0} \frac{\cos(h)-1}{h} + \cos(x) \cdot \lim_{h\to 0} \frac{\sin(h)}{h} && \text{Limit Laws} \\ &= \cos(x). && \lim_{h\to 0} \frac{\sin(h)}{h} = 1, \quad \lim_{h\to 0} \frac{\cos(x)-1}{x} = 0 \end{aligned}$$

1.5.2 Using the Basic Properties to Evaluate Derivatives

Problem 1.5. ($\star$) Using the formula for the derivative of the inverse, find the derivative of e^x.

Solution 1.5. We will use the formula

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f(x))} = \frac{1}{(\frac{d}{dx} f \circ f^{-1})(x)}$$

with $f(x) = \ln(x)$. Since $f^{-1}(x) = e^x$, and $\frac{d}{dx}\ln(x) = \frac{1}{x}$, the formula implies

$$\frac{d}{dx} e^x = \frac{1}{\frac{1}{e^x}} = e^x.$$

Problem 1.6. ($\star\star$) Given the curve $y = (4x-5)^3(x^2+5x-5)^7$, find the tangent and normal line to the curve when $x = 1$.

Solution 1.6. We have

$$\begin{aligned} &\frac{d}{dx}(4x-5)^3(x^2+5x-5)^7 \\ &= \frac{d}{dx}(4x-5)^3 \cdot (x^2+5x-5)^7 + (4x-5)^3 \cdot \frac{d}{dx}(x^2+5x-5)^7 && \text{product rule} \\ &= 3(4x-5)^2 \cdot 4 \cdot (x^2+5x-5)^7 + (4x-5)^3 \cdot 7 \cdot (x^2+5x-5)^6(2x+5). && \text{chain rule} \end{aligned}$$

Therefore,

$$f'(1) = 3 \cdot 4 \cdot (4-5)^2 \cdot (1^2+5-5)^7 + (4-5)^3 \cdot 7(1^2+5-5)^6(2+5) = 12 - 49 = -37.$$

Since $f(1) = -1$, using the formula we have that

$$y_{tangent} = f'(1)(x-1) + f(1) = -37(x-1) - 1 = -37x + 36$$

and

$$y_{normal} = -\frac{1}{f'(1)}(x-1) + f(1) = \frac{1}{37}(x-1) - 1 = \frac{1}{37}x - \frac{38}{37}.$$

Problem 1.7. ($\star$) Compute the derivative of

$$f(x) = e^{\frac{x}{x-1}}.$$

Solution 1.7. We have

$$\begin{aligned}\frac{d}{dx}e^{\frac{x}{x-1}} &= e^{\frac{x}{x-1}} \cdot \frac{d}{dx}\frac{x}{x-1} && \text{chain rule}\\ &= e^{\frac{x}{x-1}} \cdot \frac{(x-1)-x}{(x-1)^2} && \text{quotient rule}\\ &= \frac{-e^{\frac{x}{x-1}}}{(x-1)^2}.\end{aligned}$$

Problem 1.8. ($\star$) Compute the derivative of

$$f(x) = \frac{\cos(x)}{1+\tan(x)}.$$

Solution 1.8. We have

$$\begin{aligned}\frac{d}{dx}\frac{\cos(x)}{1+\tan(x)} &= \frac{-(1+\tan(x))\sin(x) - \sec^2(x)\cos(x)}{(1+\tan(x))^2} && \text{quotient rule}\\ &= -\frac{(1+\tan(x))\sin(x)+\sec(x)}{(1+\tan(x))^2}.\end{aligned}$$

Problem 1.9. ($\star\star$) Compute the second derivative of

$$f(x) = x\sec(x).$$

Solution 1.9. We have

$$\begin{aligned}\frac{d}{dx}x\sec(x) &= \sec(x) + x\sec(x)\tan(x) && \text{product rule}\\ &= \sec(x)(1+x\tan(x)).\end{aligned}$$

Taking the derivative again, we have

$$\begin{aligned}\frac{d^2}{dx^2}x\sec(x) &= \frac{d}{dx}\sec(x)(1+x\tan(x))\\ &= \sec(x)\tan(x)(1+x\tan(x)) + \sec(x)(\tan(x)+x\sec^2(x)) && \text{product rule}\\ &= x\sec(x)\tan^2(x) + 2\sec(x)\tan(x) + x\sec^3(x).\end{aligned}$$

Problem 1.10. ($\star\star$) Find the equation of the line(s) passing through the point $(-2,2)$ and tangent to the function $f(x) = x^3 - x$.

Solution 1.10. Since $f'(x) = 3x^2 - 1$, the equation of the tangent line when $x = a$ is given by

$$y = f(a) + f'(a)(x-a) = a^3 - a + (3a^2-1)(x-a).$$

If this line passes through the point $(-2,2)$, then the value $x = -2$ and $y = 2$ must lie on the curve,

$$2 = a^3 - a + (3a^2-1)(-2-a).$$

Solving for a implies that

$$2 = a^3 - a - 6a^2 + 2 - 3a^3 + a \implies a^2(a+3) = 0 \implies a = 0, -3.$$

Therefore, the equations of the tangent lines are

$$y = f(0) + f'(0)(x-0) = -x \quad \text{and} \quad y = f(-3) + f'(-3)(x+3) = 26x + 54.$$