1 Limits

1.1 Intuitive Definitions of Limits

We use the following notation to describe the limiting behavior of functions.

1. (Limit of a Function) A *limit* is written as

$$\lim_{x \to a} f(x) = L \quad \text{“the limit of } f(x) \text{ as } x \text{ approaches } a \text{ is } L\text{”.}$$

   This means f gets arbitrarily close to L whenever x is sufficiently close but not equal to a.

2. (One-sided limit) A *right one-sided limit* is written as

$$\lim_{x \to a^+} f(x) = L \quad \text{“the limit of } f(x) \text{ as } x \text{ approaches } a \text{ from the right is } L\text{”.}$$

   This means f gets arbitrarily close to L whenever x is sufficiently close but strictly greater than a.

   Similarly, a *left one-sided limit* is written as

$$\lim_{x \to a^-} f(x) = L \quad \text{“the limit of } f(x) \text{ as } x \text{ approaches } a \text{ from the left is } L\text{”.}$$

   This means f gets arbitrarily close to L whenever x is sufficiently close but strictly less than a.

3. (Limit at Infinity) A *limit at infinity* is written as

$$\lim_{x \to \infty} f(x) = L \quad \text{“the limit of } f(x) \text{ at infinity is } L\text{”.}$$

   This means f gets arbitrarily close to L whenever x is sufficiently large.

   Similarly, a *limit at negative infinity* is written as

$$\lim_{x \to -\infty} f(x) = L \quad \text{“the limit of } f(x) \text{ at negative infinity is } L\text{”.}$$

   This means f gets arbitrarily close to L whenever x is sufficiently small.

4. (Infinite Limit) If $L = +\infty$ in the definitions above, then it means that f gets arbitrarily large. Likewise, if $L = -\infty$ then we mean that f gets arbitrarily small.

1.2 Special Limits

We have the following special limits

1. $$\lim_{x \to 0} \frac{\sin x}{x} = 1,$$

2. $$\lim_{x \to 0} \frac{\cos x - 1}{x} = 0,$$

3. $$\lim_{x \to 0} (1 + ax)^{\frac{1}{x}} = e^a \text{ or } \lim_{n \to \pm\infty} \Big(1 + \frac{a}{n}\Big)^n = e^a,$$

4. $$\lim_{x \to 0} \frac{e^x - 1}{x} = 1.$$

1.3 Limit Laws

Suppose that

$$\lim_{x\to a} f(x) \text{ and } \lim_{x\to a} g(x) \text{ exists.}$$

We can use the following properties to compute the limits of complicated functions.

1. Sums of Functions:
$$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$$

2. Products of Functions:
$$\lim_{x\to a}[f(x)\cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x).$$

3. Direct Substitution: If f is continuous at a, then
$$\lim_{x\to a} f(x) = f(a).$$

4. Composition Theorem: If $f(x)$ is continuous at L and $\lim_{x\to a} g(x) = L$, then
$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(L).$$

5. Squeeze Theorem: If $g(x) \le f(x) \le h(x)$ when x is near a (except possibly at a) and
$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$$
then
$$\lim_{x\to a} f(x) = L.$$

Remark: The functions introduced in Week 1, i.e. polynomials, exponentials, trigonometric functions, hyperbolic functions, and their inverses, etc are continuous at every number in their domains.

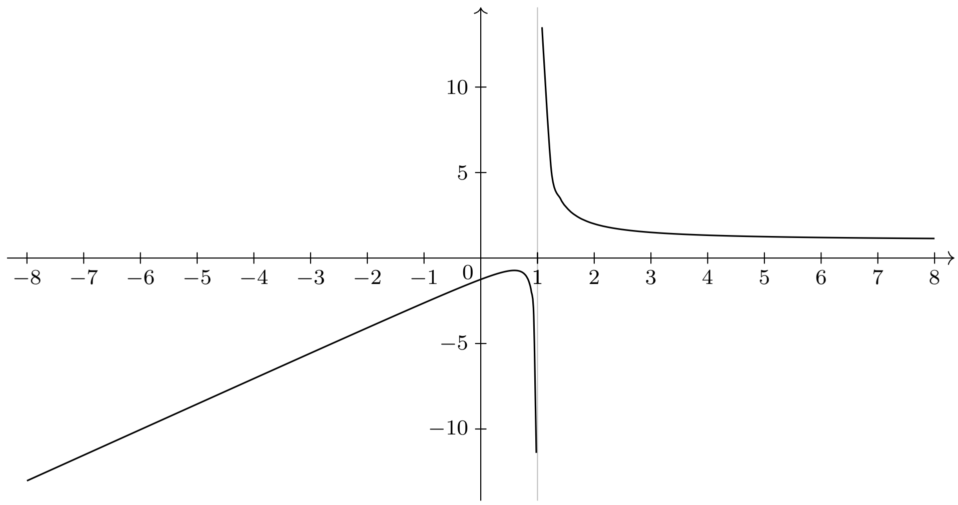
1.4 Infinite Limits and Graphs

Limits at infinity or approaching infinity tell us a lot about the shape of our graph. Consider the curve $y = f(x)$, there are three types of typical behavior described by these limits

1. **Vertical Asymptote**: A vertical line $x = a$ is called a *vertical asymptote* of the curve $y = f(x)$ if at least one of the following occurs,
$$\lim_{x\to a^-} f(x) = \pm\infty, \text{ or } \lim_{x\to a^+} f(x) = \pm\infty, \text{ or } \lim_{x\to a} f(x) = \pm\infty.$$
A curve may have multiple vertical asymptotes.

2. **Horizontal Asymptote**: A horizontal line $y = L$ is called a *horizontal asymptote* of the curve $y = f(x)$ if
$$\lim_{x\to\infty}[f(x) - L] = 0, \text{ or } \lim_{x\to-\infty}[f(x) - L] = 0.$$
Note that a curve may have 0, 1 or 2 horizontal asymptotes.

3. **Slant Asymptote**: An oblique line $y = mx + b$ (we require $m \ne 0$) is called a *slant asymptote* of the curve $y = f(x)$ if
$$\lim_{x\to\infty}[f(x) - (mx+b)] = 0 \text{ or } \lim_{x\to-\infty}[f(x) - (mx+b)] = 0.$$
Note that a curve may have 0, 1 or 2 slant asymptotes.



Example 1: The graph of

$$y = f(x) = \begin{cases} \dfrac{1}{x-1} + 1 & x > 1 \\ \dfrac{3}{2} \cdot x - 1 + \dfrac{1}{4(x-1)} & x < 1 \end{cases}$$

is displayed above. This graph has a vertical asymptote at $x = 1$, a horizontal asymptote $y = 1$ as x approaches ∞, and a slant asymptote $y = \frac{3}{2}x - 1$ as x approaches $-\infty$.

1.5 Example Problems

1.5.1 Showing a Limit Exists

Strategy: In this course, we do not have to use the definition of a limit to show a limit exists. Instead, it suffices to use an algebraic approach and limit laws to derive a limit. The following procedure generally works for combinations of functions we encountered in Week 1,

1. We first check $f(a)$ to figure out the form of the limit. If this gives us a number, then $f(a)$ is our limit and there is nothing more to do (provided that we are dealing with the composition of continuous functions).

2. If we have an indeterminate form, $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 0^0, 1^∞, ∞^0, etc then we simplify our function by factoring, collecting like terms, rationalizing, etc. The goal is to remove the "bad point" that makes our behavior indeterminate.

3. After simplifying our function, we evaluate our simplified function at a. If it gives us a number, then that is our limit.

Useful Formulas: When factoring, the following formulas may be useful

1. Difference of Powers: For $a, b \in \mathbb{R}$ and $n \geq 2$,

$$(a^n - b^n) = (a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}).$$

   There are n terms in the second part of the factored equation. For example, if $n = 4$, then

$$(x^4 - y^4) = (x - y)(x^3 + x^2y + xy^2 + y^3).$$

2. Quadratic Formulas: For $a \neq 0$, the equation $ax^2 + bx + c = 0$ has solution(s)

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Problem 1. ($\star$) Determine the following limit

$$\lim_{x \to 4} \frac{x^2 + x - 4}{x + 4}.$$

Solution 1. We first try to substitute $x = 4$ into our function

$$\frac{x^2 + x - 4}{x + 4}\Big|_{x=4} = \frac{4^2 + 4 - 4}{4 + 4} = 2.$$

This is not an indeterminate form, so we can conclude

$$\lim_{x \to 4} \frac{x^2 + x - 4}{x + 4} = 2.$$

Problem 2. ($\star$) Determine the following limit

$$\lim_{x \to -1} \frac{x^2 - 1}{x + 1}.$$

Solution 2. We first try to substitute $x = -1$ into our function

$$\frac{x^2 - 1}{x + 1}\Big|_{x=-1} = \frac{(-1)^2 - 1}{-1 + 1} = \frac{0}{0}.$$

This is an indeterminate form, so we attempt to simplify our function by factoring. Using the difference of squares formula to factor the numerator, we notice

$$\lim_{x \to -1} \frac{x^2 - 1}{x + 1} = \lim_{x \to -1} \frac{(x-1)(x+1)}{x + 1} = \lim_{x \to -1} (x - 1) = -2.$$

Problem 3. ($\star\star$) Determine the following limit

$$\lim_{x \to 1} \frac{x^5 - 1}{x^3 - 1}.$$

Solution 3. We first try to substitute $x = 1$ into our function

$$\frac{x^5 - 1}{x^3 - 1}\Big|_{x=1} = \frac{1^5 - 1}{1^3 - 1} = \frac{0}{0}.$$

This is an indeterminate form, so we attempt to simplify our function by factoring. Using the difference of powers formula to factor the numerator and denominator, we notice

$$\lim_{x \to 1} \frac{x^5 - 1}{x^3 - 1} = \lim_{x \to 1} \frac{(x-1)(x^4 + x^3 + x^2 + x + 1)}{(x-1)(x^2 + x + 1)} = \lim_{x \to 1} \frac{x^4 + x^3 + x^2 + x + 1}{x^2 + x + 1} = \frac{5}{3}.$$

Problem 4. (⋆⋆) Determine the following limit

$$\lim_{x\to 0} \frac{\sqrt{x^2+1}-1}{x^2}.$$

Solution 4. We first try to substitute $x = 1$ into our function

$$\frac{\sqrt{x^2+1}-1}{x^2}\Big|_{x=0} = \frac{1-1}{0} = \frac{0}{0}.$$

This is an indeterminate form, so we attempt to simplify our function by rationalizing the numerator. We notice

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{x^2+1}-1}{x^2} &= \lim_{x\to 0} \frac{\sqrt{x^2+1}-1}{x^2} \cdot \frac{\sqrt{x^2+1}+1}{\sqrt{x^2+1}+1} \\
&= \lim_{x\to 0} \frac{(\sqrt{x^2+1})^2-1}{x^2(\sqrt{x^2+1}+1)} \\
&= \lim_{x\to 0} \frac{1}{\sqrt{x^2+1}+1} \\
&= \frac{1}{2}.
\end{aligned}$$

1.5.2 Showing a One-Sided Limit Exists

Strategy: The strategy is identical as the last section. We might have to be careful with piecewise functions in these examples.

Problem 1. (⋆) Determine the following limit

$$\lim_{x\to 0^-} \frac{7|x|}{x}.$$

Solution 1. We first try to substitute $x = 0^-$ into our function

$$\frac{7|x|}{x}\Big|_{x=0^-} = \frac{7\cdot|0|}{0} = \frac{0}{0}.$$

This is an indeterminate form, so we attempt to simplify our function by canceling the numerator. Since $|x| = -x$ for $x < 0$, we have

$$\lim_{x\to 0^-} \frac{7|x|}{x} = \lim_{x\to 0^-} -\frac{7x}{x} = \lim_{x\to 0^-} -7 = -7.$$

Problem 2. (⋆⋆) Determine the following limit

$$\lim_{x\to 1^+} \Big(\frac{1}{x-1} - \frac{2}{x^2-1}\Big).$$

Solution 2. We first try to substitute $x = 1^+$ into our function

$$\Big(\frac{1}{x-1} - \frac{2}{x^2-1}\Big)\Big|_{x=1^+} = \infty - \infty.$$

This is an indeterminate form, so we attempt to simplify our function by collecting like terms. We notice

$$\begin{aligned}\lim_{x\to 1^+}\left(\frac{1}{x-1}-\frac{2}{x^2-1}\right) &= \lim_{x\to 1^+}\left(\frac{(x+1)}{(x-1)(x+1)}-\frac{2}{(x-1)(x+1)}\right)\\ &= \lim_{x\to 1^+}\frac{x-1}{(x-1)(x+1)}\\ &= \lim_{x\to 1^+}\frac{1}{x+1}\\ &= \frac{1}{2}.\end{aligned}$$

1.5.3 Showing a Limit Does not Exist

Strategy: Unless our function is exceptionally ugly, the usual way to show a limit does not exist is to show that its left and right limits are different.

Problem 1. ($\star$) Show that the following limit does not exist,

$$\lim_{x\to 3}\frac{x-3}{|x-3|}.$$

Solution 1. It suffices to show that our left and right limits approach different values. Since $|x| = x$ for $x > 0$, we have

$$\lim_{x\to 3^+}\frac{x-3}{|x-3|} = \lim_{x\to 3^+}\frac{x-3}{x-3} = \lim_{x\to 3^+} 1 = 1,$$

and since $|x| = -x$ for $x < 0$, we have

$$\lim_{x\to 3^-}\frac{x-3}{|x-3|} = \lim_{x\to 3^-}\frac{x-3}{-(x-3)} = \lim_{x\to 3^-} -1 = -1.$$

The left and right limits do not agree, so our limit does not exist.

Problem 2. ($\star\star\star$) Show that the following limit does not exist,

$$\lim_{x\to 0^+}\sin\left(\frac{1}{x}\right).$$

Solution 2. Let $t = 1/x$. We have $\lim_{x\to 0^+} f(x) = \lim_{t\to\infty} f(\frac{1}{t})$. In particular, we have

$$\lim_{x\to 0^+}\sin\left(\frac{1}{x}\right) = \lim_{t\to\infty}\sin(t)$$

which does not have a limit, because $\sin(t)$ oscillates between -1 and 1 infinitely often and does not converge to a particular value.

1.5.4 Limits of the composition of functions

If f is discontinuous at b and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x))$ may not behave nicely. These examples are to illustrate the potential dangers when taking limits with a discontinuous function on the outside. In these cases, evaluating the function at a may not give the right answer even if we do not get an indeterminate form.

Problem 1. ($\star\star$) Consider the discontinuous function

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$$

and let $g(x) = x^2$. Compute $\lim_{x\to 0} f(g(x))$.

Solution 1. We first compute the composition of the functions,

$$f(g(x)) = f(x^2) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0. \end{cases}$$

since $x^2 = 0$ if and only if $x = 0$. Therefore, we have

$$\lim_{x\to 0} f(g(x)) = 0,$$

since $f(g(x))$ is equal to 0 for all $x \neq 0$.

Note: We cannot take the limit inside function f in this problem. In this case, if we incorrectly used the property $\lim_{x\to 0} f(g(x)) = f(\lim_{x\to 0} g(x))$ then we will incorrectly conclude

$$\lim_{x\to 0} f(g(x)) = \lim_{x\to 0} f(x^2) = f(\lim_{x\to 0} x^2) = f(0) = 1.$$

The reason why we can't take the limit inside is that $f(x)$ is not continuous at 0, so the property does not hold. This is a reason why existence of the limits is not enough for the composition of functions.

Problem 2. ($\star\star$) Consider the discontinuous function

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

and let $g(x) = x^2$. Compute $\lim_{x\to 0} f(g(x))$.

Solution 2. We first compute the composition of the functions,

$$f(g(x)) = f(x^2) = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0. \end{cases}$$

since $x^2 > 0$ whenever $x \neq 0$ and $x^2 = 0$ when $x = 0$. Therefore, we have

$$\lim_{x\to 0} f(g(x)) = 1,$$

since $f(g(x))$ is equal to 1 for all $x \neq 0$.

Note: In this example, even though the the limit $\lim_{x\to 0} f(x)$ does not exist (check that the left and right limits give different values), the limit of $f(g(x))$ still exists. If we tried to take the limit inside, $\lim_{x\to 0} f(g(x)) = f(\lim_{x\to 0} g(x))$ then we would have gotten the incorrect answer, for the same reasoning as the previous problem.

Problem 3. ($\star\star$) Consider the discontinuous function

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0. \end{cases}$$

Compute $\lim_{x\to 0} f(f(x))$.

Solution 3. We first compute the composition of the functions,

$$f(f(x)) = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0. \end{cases}$$

Therefore, we have

$$\lim_{x\to 0} f(f(x)) = 1.$$

Note: In this case, if we incorrectly used the property $\lim_{x\to 0} f(g(x)) = f(\lim_{x\to 0} g(x))$ then we get

$$\lim_{x\to 0} f(f(x)) = f(\lim_{x\to 0} f(x)) = f(0) = 1,$$

which is the correct answer. This just happened by chance, and the steps to arrive at this conclusion is completely wrong without more justification. This is also the usual counter example that disproves the claim that if $\lim_{g\to b} f(g) = L$ and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = \lim_{g\to b} f(g) = L$.

1.5.5 Limits at Infinity (Horizontal Asymptotes)

Strategy: The limit as $x \to \infty$ is usually straightforward. We proceed as follows:

1. We first check $f(\infty)$ to figure out the form of the limit. If this gives us a number, then $f(\infty)$ is our limit and there is nothing more to do.
2. If our function is in an in-determinant form, then we simplify our equation until it is of the form $\frac{\infty}{\infty}$ or $\frac{0}{0}$.
3. Divide the numerator and denominator by the highest degree term, then take $x \to \infty$ and recompute the limit.

If we take the $x \to -\infty$ and there is a square root in our problem, then we have to be careful. It is usually best to make a change of variables $t = -x$, and use the fact $\lim_{x\to-\infty} f(x) = \lim_{t\to\infty} f(-t)$.

Problem 1. ($\star$) Determine the following limit

$$\lim_{x\to-\infty} e^{-x^2+2}.$$

Solution 1. We first try to substitute $-\infty$ into our function

$$e^{-x^2+2}\Big|_{x=-\infty} = e^{-\infty^2+2} = e^{-\infty} = 0$$

is not an indeterminate form, so

$$\lim_{x\to-\infty} e^{-x^2+2} = 0.$$

Problem 2. ($\star$) Determine the following limit

$$\lim_{x\to\infty} \frac{x+1}{\sqrt{x^2+2x+7}}.$$

Solution 2. We first try to substitute ∞ into our function

$$\frac{x+1}{\sqrt{x^2+2x+7}}\Big|_{x=\infty} = \frac{\infty}{\infty}.$$

This is an indeterminate form of the form $\frac{\infty}{\infty}$, so we can divide the numerator and denominator by the highest degree polynomial,

$$\lim_{x\to\infty} \frac{x+1}{\sqrt{x^2+2x+7}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{x}} \cdot \frac{x+1}{\sqrt{x^2+2x+7}} = \lim_{x\to\infty} \frac{1+\frac{1}{x}}{\sqrt{1+\frac{2}{x}+\frac{7}{x^2}}} = 1.$$

Problem 3. ($\star\star$) Determine the following limit

$$\lim_{x\to-\infty} \frac{x+1}{\sqrt{x^2+2x+7}}.$$

Solution 3. We first try to substitute $-\infty$ into our function

$$\frac{x+1}{\sqrt{x^2+2x+7}}\Big|_{x=\infty} = -\frac{\infty}{\infty}.$$

This is an indeterminate form of the form $\frac{\infty}{\infty}$, so we can divide the numerator and denominator by the highest degree polynomial. Since there is a square root in our question, we have to be careful because we cannot put a negative number inside of the square root to simplify it. Instead, we use the change of variables $t = -x$ and notice

$$\lim_{x\to-\infty} \frac{x+1}{\sqrt{x^2+2x+7}} = \lim_{t\to\infty} \frac{-t+1}{\sqrt{(-t)^2-2t+7}} = \lim_{t\to\infty} \frac{\frac{1}{t}}{\frac{1}{t}} \cdot \frac{-t+1}{\sqrt{t^2-2t+7}} = \lim_{t\to\infty} \frac{-1+\frac{1}{t}}{\sqrt{1-\frac{2}{t}+\frac{7}{t^2}}} = -1.$$

Problem 4. ($\star\star$) Determine the following limit

$$\lim_{x\to\infty} (\sqrt{9x^2+x} - 3x).$$

Solution 4. We first try to substitute ∞ into our function

$$(\sqrt{9x^2+x} - 3x)\Big|_{x=\infty} = \infty - \infty.$$

This is an indeterminate form of the form $\infty - \infty$. This form is really hard to work with, so we first rewrite our function by rationalizing the root

$$\lim_{x\to\infty} (\sqrt{9x^2+x} - 3x) = \lim_{x\to\infty} (\sqrt{9x^2+x} - 3x) \cdot \frac{\sqrt{9x^2+x}+3x}{\sqrt{9x^2+x}+3x} = \lim_{x\to\infty} \frac{x}{\sqrt{9x^2+x}+3x}.$$

It is easy to check that this limit is of the form $\frac{\infty}{\infty}$, so we can divide the numerator and denominator by the highest power and conclude

$$\lim_{x\to\infty} \frac{x}{\sqrt{9x^2+x}+3x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{x}} \frac{x}{\sqrt{9x^2+x}+3x} = \lim_{x\to\infty} \frac{1}{\sqrt{9+\frac{1}{x}}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}.$$

1.5.6 Infinite Limits (Vertical Asymptotes)

Strategy: To find the behavior of our function at a vertical asymptote, we first substitute a value of x at the asymptote, and remember the asymptotic behavior of the graphs of the standard function

$$\frac{1}{0^+} = \infty,\ \frac{1}{0^-} = -\infty,\ e^{\infty} = \infty,\ \ln(0^+) = -\infty,\ \ln(\infty) = \infty,\ \tan(\frac{\pi}{2}^-) = \infty,\ \text{etc}$$

If our answer is infinite, then we don't need to do any more work. However, if we have an indeterminate form, then we can proceed like usual to compute the limit.

If we want to find the vertical asymptotes of a combination of basic functions introduced, we should start by checking for asymptotes where $f(x)$ is undefined.

Problem 1. ($\star$) Determine the following limit

$$\lim_{x\to 1^-} \Big(\frac{2}{x} - \frac{1}{\ln(x)}\Big).$$

Solution 1. We first substitute $x = 1^-$ into our function

$$\frac{2}{x} - \frac{1}{\ln(x)}\Big|_{x=1^-} = \frac{2}{1^-} - \frac{1}{\ln(1^-)} = 2 - \frac{1}{0^-} = 2 + \infty = \infty.$$

This is not an indeterminate form, so we can conclude,

$$\lim_{x\to 1^-} \Big(\frac{2}{x} - \frac{1}{\ln(x)}\Big) = \infty.$$

Problem 2. ($\star\star$) Find the vertical asymptotes of the function

$$f(x) = \frac{x+1}{x^2-4}.$$

Solution 2. Since $f(x)$ is undefined when $x = \pm 2$, and continuous everywhere else, it suffices to check for vertical asymptotes at $x = \pm 2$. Notice that

$$\lim_{x\to 2^+} \frac{x+1}{x^2-4} = \frac{3}{0^+} = \infty \quad \text{and} \quad \lim_{x\to 2^-} \frac{x+1}{x^2-4} = \frac{3}{0^-} = -\infty$$

so there is vertical asymptote at $x = 2$. Similarly,

$$\lim_{x\to -2^+} \frac{x+1}{x^2-4} = \frac{-1}{0^-} = \infty \quad \text{and} \quad \lim_{x\to -2^-} \frac{x+1}{x^2-4} = \frac{-1}{0^+} = -\infty$$

so there is vertical asymptote at $x = -2$.

1.5.7 Special case of a limit at infinity (Slant Asymptotes)

Recall that a line $y = mx + b$ and $m \neq 0$ is called a slant asymptote if

$$\lim_{x\to\infty} [f(x) - (mx + b)] = 0 \text{ or } \lim_{x\to -\infty} [f(x) - (mx + b)] = 0.$$

We call the line $mx+b$ a slant asymptote at ∞ (or a slant asymptote at $-\infty$ if the second case occurs).

Problem 1. ($\star\star$) Find the slant asymptotes of the function

$$f(x) = \frac{x^3}{x^2+1}.$$

Solution 1. Using long division, we notice that

$$\frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}.$$

In this form, we see that $f(x)$ is almost the equation of a line $x+b$, with the intercept term $b = -\frac{x}{x^2+1}$ going to 0 as x gets large. To check that x is a slant asymptote at ∞, we have

$$\lim_{x\to\infty}\left[\frac{x^3}{x^2+1} - x\right] = \lim_{x\to\infty}\frac{x}{x^2+1} = 0$$

and similarly, to check that x is a slant asymptote at $-\infty$, we have

$$\lim_{x\to-\infty}\left[\frac{x^3}{x^2+1} - x\right] = \lim_{x\to-\infty}\frac{x}{x^2+1} = 0.$$

Problem 2. ($\star\star\star$) Find the slant asymptotes of the function

$$f(x) = \pi x + \tan^{-1}(x).$$

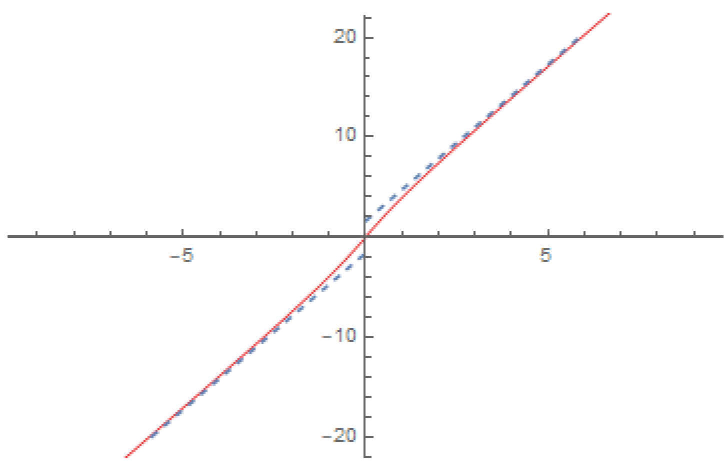
Solution 2. We see that $f(x)$ is almost the equation of a line $\pi x + b$, with the intercept term $b = \tan^{-1}(x)$ going to $\pi/2$ as x gets large and $-\pi/2$ as x gets small. To check that $\pi x + \pi/2$ is a slant asymptote at ∞, we have

$$\lim_{x\to\infty}\left[\pi x + \tan^{-1}(x) - (\pi x + \frac{\pi}{2})\right] = \lim_{x\to\infty}\tan^{-1}(x) - \frac{\pi}{2} = 0$$

and similarly, to check that $\pi x - \pi/2$ is a slant asymptote at $-\infty$, we have

$$\lim_{x\to-\infty}\left[\pi x + \tan^{-1}(x) - (\pi x - \frac{\pi}{2})\right] = \lim_{x\to-\infty}\tan^{-1}(x) + \frac{\pi}{2} = 0.$$

The graph of the $f(x)$ is in red and the slant asymptotes are plotted in the dotted blue lines,



1.5.8 Squeeze Theorem Problems

Strategy: We want to find upper and lower bounds of $f(x)$ that have the same limits as $x \to a$.

Problem 1. ($\star$) Determine the following limit

$$\lim_{x\to 0^+} x\sin\left(\frac{1}{x}\right).$$

Solution 1. We first substitute $x = 0^+$ into our function

$$x\sin\left(\frac{1}{x}\right)\Big|_{x=0^+} = 0\cdot\sin(\infty).$$

which is an indeterminate form. There is no algebraic way of simplifying the $\sin(x)$, so we use the Squeeze theorem to find an appropriate limit. Notice that the range of $\sin(x) \in [-1,1]$, so we have

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1 \text{ for all } x > 0.$$

Therefore, multiplying both sides by $x > 0$, we have

$$-x \le x\sin\left(\frac{1}{x}\right) \le x$$

Computing the limits of the lower and upper bounds we see

$$\lim_{x\to 0^+}(-x) = 0 \text{ and } \lim_{x\to 0^+} x = 0.$$

These limits are the same, so the squeeze theorem implies

$$\lim_{x\to 0^+} x\sin\left(\frac{1}{x}\right) = 0.$$

Problem 2. ($\star\star$) Determine the following limit

$$\lim_{x\to 0^+} \sqrt{x}e^{\sin(\pi/x)}.$$

Solution 2. We first substitute $x = 0^+$ into our function

$$\sqrt{x}e^{\sin(\pi/x)}\Big|_{x=0^+} = 0\cdot e^{\sin(\infty)}.$$

which is an indeterminate form. There is no algebraic way of simplifying the $e^{\sin(\pi/x)}$, so we use the Squeeze theorem to find an appropriate limit. Notice that the range of $\sin(x) \in [-1,1]$, so we have

$$-1 \le \sin\left(\frac{\pi}{x}\right) \le 1 \text{ for all } x > 0.$$

Therefore, exponentiating everything we have

$$e^{-1} \le e^{\sin(\pi/x)} \le e^1$$

and then multiplying by $\sqrt{x}$, we have

$$\sqrt{x}e^{-1} \le \sqrt{x}e^{\sin(\pi/x)} \le \sqrt{x}e^1.$$

Computing the limits of the lower and upper bounds we see

$$\lim_{x\to 0^+}\sqrt{x}e^{-1} = 0 \text{ and } \lim_{x\to 0^+}\sqrt{x}e^1 = 0.$$

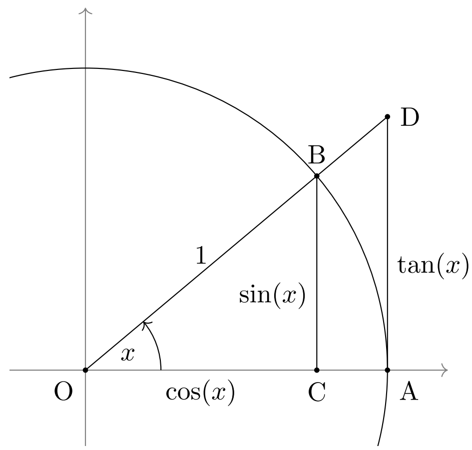
These limits are the same, so the squeeze theorem implies

$$\lim_{x\to 0^+}\sqrt{x}e^{\sin(\pi/x)} = 0.$$

Problem 3. ($\star\star\star$) Prove the basic trigonometric limit,

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

Solution 3. Let $0 < x < \frac{\pi}{2}$ and consider the picture of the unit circle:



From the picture, we see that the area of the inner triangle $\triangle OCB$ is less than the area of the sector $\sphericalangle OAB$, which is less than the area of the outer triangle $\triangle OAD$. Computing the areas explicitly using the formula for the triangle and the sector of a circle, we have

$$\frac{1}{2}\sin(x)\cos(x) \leq \frac{1}{2}x \leq \frac{1}{2}\tan(x) \Rightarrow \sin(x)\cos(x) \leq x \leq \tan(x).$$

Since $\sin(x) > 0$ on this interval, we can divide by $\sin(x)$ and conclude

$$\cos(x) \leq \frac{x}{\sin(x)} \leq \frac{1}{\cos(x)}.$$

Taking reciprocals (the direction of the inequalities reverse), we have

$$\cos(x) \leq \frac{\sin(x)}{x} \leq \frac{1}{\cos(x)}.$$

Computing the limits as $x \to 0^+$ of the upper and lower bounds, we see

$$\lim_{x\to 0^+} \frac{1}{\cos(x)} = 1, \text{ and } \lim_{x\to 0^+} \cos(x) = 1.$$

These limits are the same, so the squeeze theorem implies

$$\lim_{x\to 0^+} \frac{\sin(x)}{x} = 1.$$

To compute the other one-sided limit, notice that $\frac{\sin(x)}{x}$ is even, so $\lim_{x\to 0^-} \frac{\sin(x)}{x} = 1$ by symmetry.

Problem 4. ($\star\star$) Determine the following limit

$$\lim_{x\to 0^+} \frac{\sin(x^2)+x}{\sqrt{x^4+4x^2}}.$$

Hint: Use the fact $|\sin(\theta)| \leq |\theta|$ for all $\theta \in \mathbb{R}$.

Solution 4. We first substitute $x = 0^+$ into our function

$$\frac{\sin(x^2)+x}{\sqrt{x^4+4x^2}}\Big|_{x=0^+} = \frac{0}{0}.$$

which is an indeterminate form. For $\theta \geq 0$, we have the bounds $-\theta \leq \sin(\theta) \leq \theta$ (we can prove this bound later in the course using the mean value theorem). Taking $\theta = x^2$, for $x > 0$ we have

$$-x^2 \leq \sin(x^2) \leq x^2.$$

Therefore, we can add x and divide by $\sqrt{x^4+4x^2}$ to conclude

$$\frac{-x^2+x}{\sqrt{x^4+4x^2}} \leq \frac{\sin(x^2)+x}{\sqrt{x^4+4x^2}} \leq \frac{x^2+x}{\sqrt{x^4+4x^2}}.$$

Computing the limits of the lower and upper bounds, we see

$$\lim_{x\to 0^+} \frac{-x^2+x}{\sqrt{x^4+4x^2}} = \lim_{x\to 0^+} \frac{-x+1}{\sqrt{x^2+4}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$$

and

$$\lim_{x\to 0^+} \frac{x^2+x}{\sqrt{x^4+4x^2}} = \lim_{x\to 0^+} \frac{x+1}{\sqrt{x^2+4}} = \frac{1}{\sqrt{4}} = \frac{1}{2}.$$

These limits are the same, so the squeeze theorem implies

$$\lim_{x\to 0^+} \frac{\sin(x^2)+x}{\sqrt{x^4+4x^2}} = \frac{1}{2}.$$

Note: The identity $0 \leq \sin(\theta) \leq \theta$ for $\theta > 0$ can also be used to derive the same limit. However, one can check that the usual bounds $-1 \leq \sin(x) \leq 1$ will not work in this problem, so we have to use a sharper bound for $\sin(x)$.

1.5.9 Using Special Limits

Problem 1. ($\star\star$) Determine the following limit

$$\lim_{x\to 0^+} \frac{\sin(x^2)+x}{\sqrt{x^4+4x^2}}.$$

Solution 1. Instead of using the squeeze theorem like in the last section, we can use the identity $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. In this case, if we multiply the numerator and denominator by x, we have

$$\lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{1}{x}} \cdot \frac{\sin(x^2)+x}{\sqrt{x^4+4x^2}} = \lim_{x\to 0^+} \frac{\frac{\sin(x^2)}{x}+1}{\sqrt{x^2+4}} = \lim_{x\to 0^+} \frac{\frac{\sin(x^2)}{x^2}\cdot x+1}{\sqrt{x^2+4}} = \frac{1}{\sqrt{4}} = \frac{1}{2}.$$

Problem 2. (⋆⋆) Determine the following limit

$$\lim_{x\to 0} \frac{x\sin(3x)\tan(x)}{\sin^3(9x)}.$$

Solution 2. We can use the identity $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ to solve this problem. Since our functions are continuous, we can use the limit composition theorem to conclude for all $a \neq 0$,

$$\lim_{x\to 0} \frac{\sin(ax)}{x} = a \lim_{x\to 0} \frac{\sin(ax)}{ax} = a \cdot \lim_{t\to 0} \frac{\sin(t)}{t} = a \text{ and } \lim_{x\to 0} \frac{x}{\sin(ax)} = \lim_{x\to 0} \Big(\frac{\sin(ax)}{x}\Big)^{-1} = \frac{1}{a}$$

where we used the fact that $x \to 0$ is the same as $t = ax \to 0$. Therefore,

$$\begin{aligned}
\lim_{x\to 0} \frac{x\sin(3x)\tan(x)}{\sin^3(9x)} &= \lim_{x\to 0} \frac{x\sin(3x)\sin(x)}{\sin^3(9x)\cos(x)} \\
&= \lim_{x\to 0} \Big(\frac{x}{\sin(9x)} \cdot \frac{\sin(3x)}{\sin(9x)} \cdot \frac{\sin(x)}{\sin(9x)} \cdot \frac{1}{\cos(x)}\Big) \\
&= \lim_{x\to 0} \Big(\frac{x}{\sin(9x)} \cdot \frac{\sin(3x)}{x} \cdot \frac{x}{\sin(9x)} \cdot \frac{\sin(x)}{x} \cdot \frac{x}{\sin(9x)} \cdot \frac{1}{\cos(x)}\Big) \\
&= \frac{1}{9} \cdot 3 \cdot \frac{1}{9} \cdot 1 \cdot \frac{1}{9} \cdot 1 \qquad \lim_{x\to 0} \frac{\sin(ax)}{x} = a \\
&= \frac{1}{243}.
\end{aligned}$$

Problem 3. (⋆⋆) Determine the following limit

$$\lim_{x\to 0} \frac{1-\cos(3x)}{\sin(2x)}.$$

Solution 3. We can use the identity $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ and $\lim_{x\to 0} \frac{\cos(x)-1}{x} = 0$ to solve this problem. We have,

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos(3x)}{\sin(2x)} &= \lim_{x\to 0} \Big(-\frac{\cos(3x)-1}{x} \cdot \frac{x}{\sin(2x)}\Big) \\
&= \lim_{x\to 0} \Big(-\frac{3}{2} \cdot \frac{\cos(3x)-1}{3x} \cdot \frac{2x}{\sin(2x)}\Big) \\
&= -\frac{3}{2} \cdot 0 \cdot \frac{1}{2} \qquad \lim_{x\to 0} \frac{\sin(ax)}{ax} = 1, \quad \lim_{x\to 0} \frac{\cos(ax)-1}{ax} = 0 \\
&= 0.
\end{aligned}$$