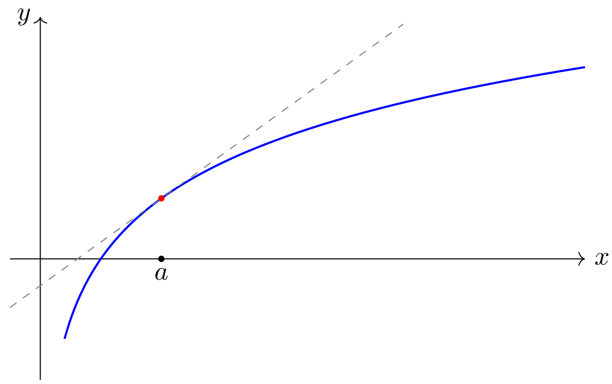
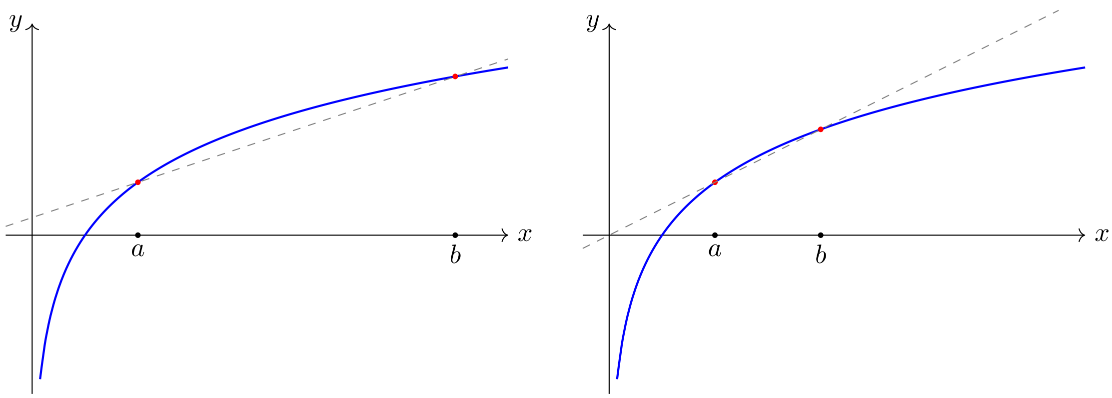
# 1 The Tangent Line Problem and the Derivative

**Question:** *Given the graph of a function $y = f(x)$, what is the slope of the curve at the point $(a, f(a))$?*



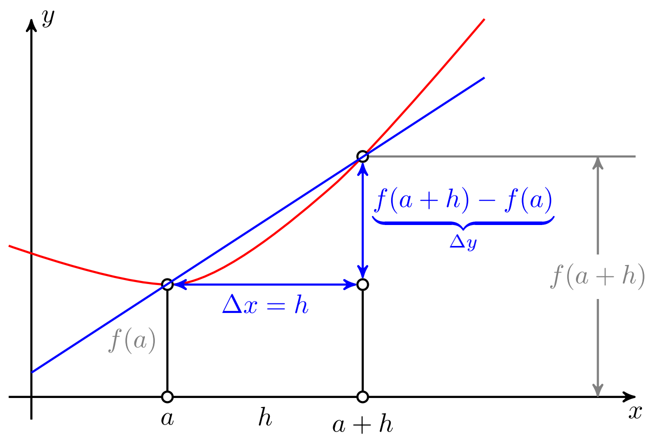

Our strategy is to approximate the slope by a limit of secant lines between points $(a, f(a))$ and $(b, f(b))$. The approximation improves as $b$ gets closer and closer to $a$.




**Definition 1.** The *difference quotient* is the slope of a secant line approximation for $y = f(x)$ between points $(a, f(a))$ and $(a+h, f(a+h))$ for $h > 0$ and is given by the formula

$$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}.$$

The slope of the tangent line is approximated by the difference quotient. The secant line approximation can be visualized below



**Definition 2.** The slope $f'(a)$ of the tangent line to $f(x)$ at point $a$ is given by

$$f'(a) = \frac{dy}{dx}\Big|_{x=a} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

This is the limit of secant of secant lines between the points $(a, f(a))$ and $(a+h, f(a+h))$ as $h \to 0$. If the number $f'(a)$ exists, then we say $f$ is *differentiable* at $a$ and we call the quantity $f'(a)$ the *derivative* of $f$ at $a$.

## 1.1 Application to Velocity

Let $s(t)$ be the position of a particle at time $t$. In this context, Definition 1 and Definition 2 have the following interpretations

1. Secant Line: The *average velocity* $v_{\text{av}}$ of the particle is given by the secant line approximation of the function $s(t)$ on the interval $a \leq t \leq b$,

$$v_{\text{av}} = \frac{s(b) - s(a)}{b - a}$$

2. Tangent Line: The *instantaneous velocity* $v_{\text{inst}}$ is the tangent line of the function $s(t)$ at the point $x = a$

$$v_{\text{inst}} = \lim_{h\to 0} \frac{s(a+h) - s(a)}{h}$$

## 1.2 Example Problems

*Useful Formulas:* The equation of a tangent line approximation of the function $f$ at the point $x = a$ is given by

$$\frac{y - f(a)}{x - a} = f'(a).$$

**Problem 1.1.** ($\star$) Let $f(x) = x^2 - 2$, find the secant line between the points $(1, f(1))$ and $(4, f(4))$

**Solution 1.1.** Taking $a = 1$ and $h = 3$ in our formula, we have

$$\frac{\Delta y}{\Delta x} = \frac{f(4) - f(1)}{4 - 1} = \frac{14 + 1}{3} = 5.$$

**Problem 1.2.** ($\star$) Suppose that the position of a particle moving horizontally on the $x$-axis is given by $s(t) = t^3 - 1$ for $t \in [0, 10]$.

a) Find the average velocity of the object on the time interval $[0, 5]$.

b) Find the instantaneous velocity at time $t = 1$.

**Solution 1.2.**

*Part a)* Taking $a = 0$ and $h = 5$ in our formula, the average velocity is given by

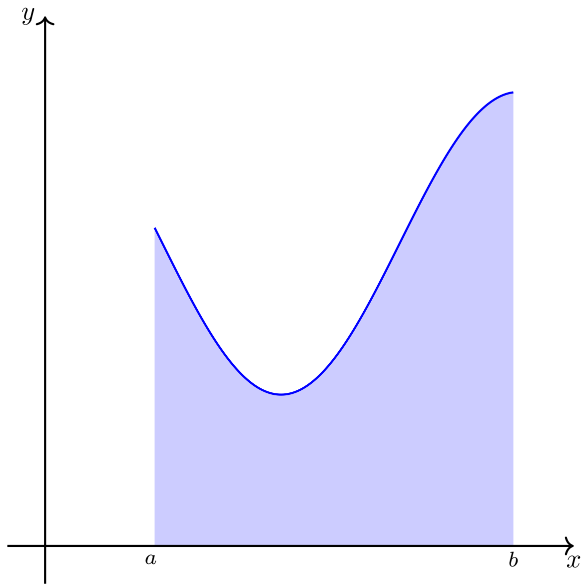
$$\frac{\Delta x}{\Delta t} = \frac{s(5) - s(0)}{5} = \frac{(5^3 - 1) - (-1)}{5} = 25.$$

*Part b)* The instantaneous velocity is given by

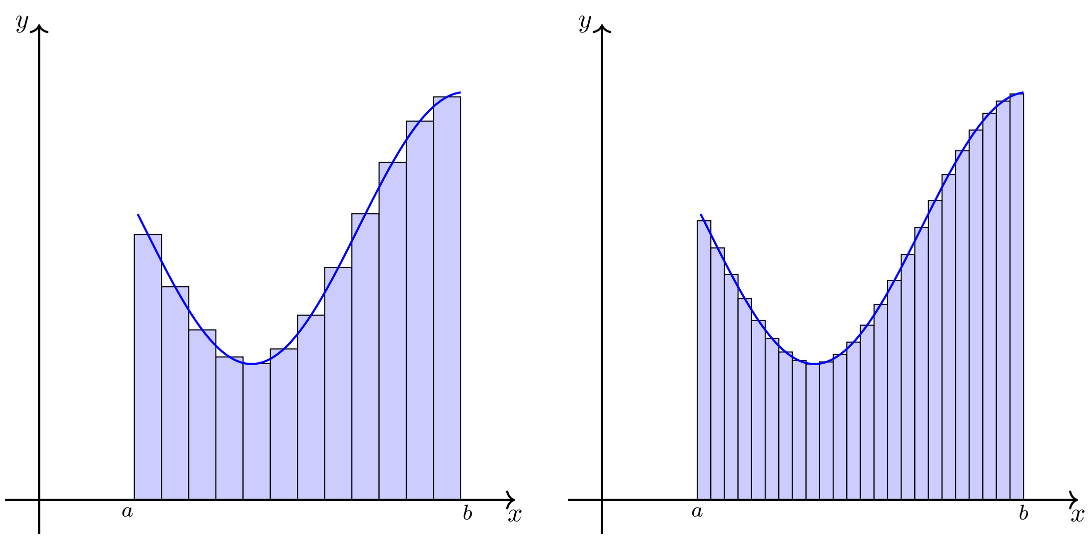
$$\frac{ds}{dt}\Big|_{t=1} = \lim_{h\to 0} \frac{s(1+h) - s(1)}{h} = \lim_{h\to 0} \frac{(1+h)^3 - 1 - 0}{h} = \lim_{h\to 0} \frac{h^3 + 3h^2 + 3h + 1 - 1}{h} = 3.$$

## 2 The Area Problem and the Definite Integral

**Question:** *Given the graph of a function $y = f(x)$, what is the net area (the area above the $x$-axis and under the curve $f$ minus the area below the $x$-axis and above the curve of $f$) of the graph between the points $a$ and $b$?*




Our strategy is to divide the region $[a, b]$ into $n$ subintervals and approximate the area by a limit of rectangles approximating our function. The approximation improves by taking $n$ larger and larger.




**Definition 3.** The *Riemann sum* approximation of $\int_a^b f(x)\,dx$ on the interval $[a, b]$ with $n$ uniform subintervals is given by

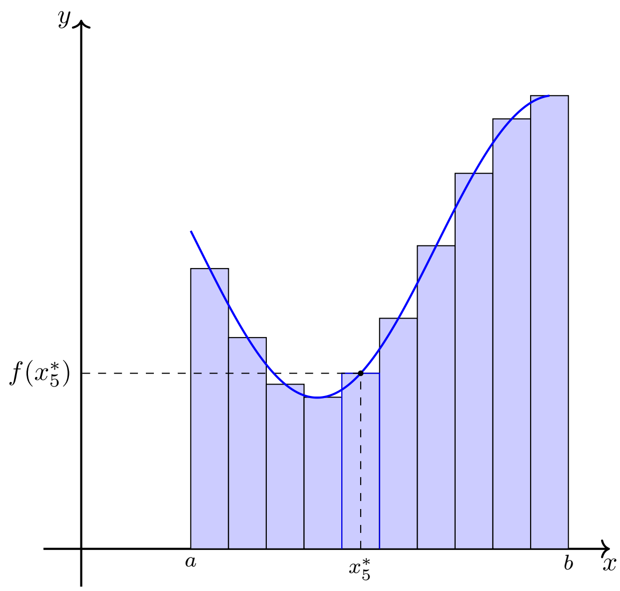
$$S_n = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

where $\Delta x = \frac{(b-a)}{n}$ and $x_i^* \in [a + (i-1)\Delta x, a + i\Delta x]$. The net area under the graph $f$ is approximated by the Riemann sum.

**Remark:** We usually sample our function $f$ at the right endpoint, midpoint, or left endpoint of each interval. The formula for $x_i$ in each of these cases is given by:

1. Right Riemann Sum: $x_i^* = a + i\Delta x$
2. Midpoint Riemann Sum: $x_i^* = a + (i - \frac{1}{2})\Delta x$
3. Left Riemann Sum: $x_i^* = a + (i - 1)\Delta x$

The midpoint approximation can be visualized below




**Definition 4.** The net area of the graph $f$ on the interval $[a, b]$ is given by the definite integral of $f(x)$ on $[a, b]$. We call the quantity $\int_a^b f(x)\,dx$ the *definite integral* of $f$ on $[a, b]$, and it is defined by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

where $\Delta x = \frac{(b-a)}{n}$ and $x_i^* \in [a + (i-1)\Delta x, a + i\Delta x]$. This is the limit of Riemann sum approximations as $n \to \infty$. If the number $\int_a^b f(x)\,dx$ exists and is identical for all choices of samples $x_i^*$, then we say $f$ is *integrable* on $[a, b]$.

## 2.1 Accuracy of Riemann Sum Approximations

Without doing any computations, we can determine if the Riemann sums are over or under approximations by looking at the shape of the curve we want to estimate the area of:

| $f(x)$ | Left | Right | Midpoint |
|---|---|---|---|
| Increasing | Under | Over | ? |
| Decreasing | Over | Under | ? |
| Convex | ? | ? | Under |
| Concave | ? | ? | Over |

For example, the table says that if $f(x)$ is increasing on $[a, b]$, then the left Riemann sum is an under approximation of the definite integral, and the right Riemann sum is an over approximation of the definite integral. The fact $f$ is increasing does not tell us enough to determine if the midpoint is an over or under approximation in general.

## 2.2 Application to Velocity

Let $v(t)$ be the velocity of a particle at time $t$. In this context, Definition 4 has the following interpretations

1. Definite Integral of $|f|$: The *distance traveled* by the particle is given by the definite integral of $|v|$ on the interval $a \leq t \leq b$, which is given explicitly by the formula
$$\int_a^b |v(t)|\, dt.$$

2. Definite Integral of $f$: The *net distance traveled* (or *displacement*) $d_v$ of the particle is given by the definite integral of $|v|$ on the interval $a \leq t \leq b$, which is given explicitly by the formula
$$\int_a^b v(t)\, dt.$$

## 2.3 Example Problems

*Useful Formulas:* The following formulas for the partial sums of a number will be useful to compute the Riemann Sums of certain functions

1. Sum of first $n$ constants:
$$\sum_{i=1}^{n} 1 = n. \tag{1}$$

2. Sum of first $n$ integers:
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}. \tag{2}$$

3. Sum of first $n$ squares:
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}. \tag{3}$$

4. Sum of first $n$ cubes:
$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2. \tag{4}$$

5. Geometric series:
$$\sum_{i=1}^{n} r^i = r\left(\frac{1-r^n}{1-r}\right). \tag{5}$$

**Problem 2.1.** ($\star$) Approximate the value of $\int_1^2 \ln(x)\, dx$ by using a left endpoint Riemann sum and 4 uniform subintervals.

**Solution 2.1.** We take $f(x) = \ln(x)$, $a = 1$, $b = 2$, and $n = 4$ in Definition 3. Since we are sampling at the left endpoints, we choose $x_i = 1 + (i-1)\Delta x$ where $\Delta x = \frac{b-a}{n} = \frac{1}{4}$. Using our formula, we have

$$\begin{aligned} S_4 = \sum_{i=1}^{4} f\big(1 + (i-1)\Delta x\big)\Delta x &= \sum_{i=1}^{4} f\left(1 + \frac{i-1}{4}\right)\frac{1}{4} \\ &= \frac{1}{4}\sum_{i=1}^{4} \ln\left(1 + \frac{i-1}{4}\right) \\ &= \frac{1}{4}\Big(\ln(1) + \ln(1.25) + \ln(1.5) + \ln(1.75)\Big) \approx 0.2970\ldots \end{aligned}$$

**Problem 2.2.** ($\star$) Approximate the area under the curve $y = 2x$ above the $x$-axis on the interval $[0, 10]$ using 10 uniform subintervals and samples at the right endpoint of each interval.

**Solution 2.2.** We take $f(x) = 2x$, $a = 0$, $b = 10$, and $n = 10$ in Definition 3. Since we are sampling at the right endpoints, we choose $x_i^* = i\Delta x$ where $\Delta x = \frac{b-a}{n} = 1$. Using our formula, we have

$$\begin{aligned} S_{10} = \sum_{i=1}^{10} f(i\Delta x)\Delta x = \sum_{i=1}^{10} 2i &= 2\sum_{i=1}^{10} i \\ &= 2\frac{10(10+1)}{2} = 110. \qquad \text{since } \sum_{i=1}^{n} i = \frac{n(n+1)}{2}. \end{aligned}$$

**Problem 2.3.** ($\star\star$) Approximate the area under the curve $y = 2x$ above the $x$-axis on the interval $[0, 10]$ using $n$ uniform subintervals and samples at the right endpoint of each interval. What does the sum converge to when we take $n \to \infty$?

**Solution 2.3.** We take $f(x) = 2x$, $a = 0$, $b = 10$, with variable $n$ in Definition 3. Since we are sampling at the right endpoints, we choose $x_i^* = i\Delta x$ where $\Delta x = \frac{10-0}{n} = \frac{10}{n}$. Using our formula, we have

$$\begin{aligned} S_n = \sum_{i=1}^{n} f\big(i\Delta x\big)\Delta x = \sum_{i=1}^{n} 2\cdot\frac{10i}{n}\cdot\frac{10}{n} &= \frac{200}{n^2}\sum_{i=1}^{n} i \\ &= \frac{200}{n^2}\cdot\frac{n(n+1)}{2} = 100\cdot\frac{n+1}{n}. \qquad \text{since } \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \end{aligned}$$

As $n \to \infty$, we have

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 100\cdot\frac{n+1}{n} = 100.$$

**Remark.** The final answer is the same as

$$\int_0^{10} 2x\,dx = x^2\Big|_{x=0}^{x=10} = 100.$$

This is also the same as the area of a triangle with base 10 and height 20.

**Problem 2.4.** ($\star\star$) Approximate the area under the curve $y = x^2$ above the $x$-axis on the interval $[0, 1]$ using 100 uniform subintervals and samples at the left endpoint of each interval.

**Solution 2.4.** We take $f(x) = x^2$, $a = 0$, $b = 1$, and $n = 100$ in Definition 3. Since we are sampling at the left endpoints, we choose $x_i^* = (i-1)\Delta x$ where $\Delta x = \frac{1}{100}$. Using our formula, we have

$$\begin{aligned} S_{100} = \sum_{i=1}^{100} f\big((i-1)\Delta x\big)\Delta x &= \sum_{i=1}^{100}\left(\frac{i-1}{100}\right)^2\frac{1}{100} \\ &= \frac{1}{100^3}\sum_{i=1}^{100}(i-1)^2 \\ &= \frac{1}{100^3}\sum_{i=0}^{99} j^2 \qquad \text{by reindexing } j = i-1 \\ &= \frac{1}{100^3}\cdot\frac{99(100)(199)}{6} \qquad \text{since } \sum_{j=0}^{n} j^2 = \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6} \\ &= 0.32835. \end{aligned}$$

**Problem 2.5.** ($\star\star$) Approximate the area under the curve $y = 3^x$ on the interval $[0, 1]$ using 1000 uniform subintervals and samples at the left endpoint. Is the approximation an over or under approximation of the area?

**Solution 2.5.** We take $f(x) = 3^x$, $a = 0$, $b = 1$, and $n = 1000$ in Definition 3. Since we are sampling at the left endpoints, we choose $x_i^* = (i-1)\Delta x$ where $\Delta x = \frac{1}{1000}$. Using the formula for the geometric series implies that

$$S_{1000} = \sum_{i=1}^{1000} f(x_i^*)\Delta x = \frac{1}{1000}\sum_{i=1}^{1000} 3^{\frac{i-1}{1000}} = \frac{1}{1000}\cdot\frac{1}{3^{\frac{1}{1000}}}\sum_{i=1}^{1000}\left(3^{\frac{1}{1000}}\right)^i = \frac{1}{1000}\cdot\left(\frac{1-3^{\frac{1000}{1000}}}{1-3^{\frac{1}{1000}}}\right) \approx 1.81948.$$

Since $f'(x) = \ln(3)\cdot 3^x \geq 0$ on $[0, 1]$, our function is increasing, and therefore the Riemann sum is an under approximation of the area.

**Remark.** The approximate area is very close to the real area,

$$\int_0^1 3^x\,dx = \frac{3^x}{\ln(3)}\bigg|_{x=0}^{x=1} = \frac{3-1}{\ln(3)} \approx 1.8205.$$

**Problem 2.6.** ($\star\star\star$) Approximate the area under the curve $y = x^2$ above the $x$-axis on the interval $[0, 1]$ using $n$ uniform subintervals and samples at the midpoint of each interval. What does the sum converge to when we take $n \to \infty$?

**Solution 2.6.** We take $f(x) = x^2$, $a = 0$, $b = 1$, with variable $n$ in Definition 3. Since we are sampling at the midpoints of the intervals, we choose $x_i^* = (i - \frac{1}{2})\Delta x$ where $\Delta x = \frac{1}{n}$. Using our formula, we have

$$\begin{aligned}
S_n &= \sum_{i=1}^{n} f\Big(\Big(i-\frac{1}{2}\Big)\Delta x\Big)\Delta x \\
&= \sum_{i=1}^{n}\Big(\frac{2i-1}{2n}\Big)^2\frac{1}{n} \\
&= \frac{1}{4n^3}\sum_{i=1}^{n}(2i-1)^2 \\
&= \frac{1}{4n^3}\sum_{i=1}^{n}(4i^2-4i+1) \\
&= \frac{1}{4n^3}\left(4\sum_{i=1}^{n} i^2 - 4\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right) \\
&= \frac{1}{4n^3}\left(4\cdot\frac{n(n+1)(2n+1)}{6} - 4\cdot\frac{n(n+1)}{2} + n\right) \qquad \text{using formulas (1), (2), (3)} \\
&= \frac{n(n+1)(2n+1)}{6n^3} - \frac{(n+1)}{2n^2} + \frac{1}{4n^2}.
\end{aligned}$$

As $n \to \infty$, we have

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{n(n+1)(2n+1)}{6n^3} - \frac{(n+1)}{2n^2} + \frac{1}{4n^2}\right) = \frac{1}{3}.$$

**Remark.** The final answer is the same as

$$\int_0^1 x^2\,dx = \frac{x^3}{3}\bigg|_{x=0}^{x=1} = \frac{1}{3}.$$