1 Using Integration to Find Arc Lengths and Surface Areas

1.1 Arc Length

Formula: If $f'(x)$ is continuous on $[a, b]$, then the *arc length of the curve* $y = f(x)$ on the interval $[a, b]$ is given by

$$s = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

Remark. We can remember this formula using the differential notation

$$(ds)^2 = (dx)^2 + (dy)^2 \Leftrightarrow ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \Leftrightarrow ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

The arc length s can be recovered by integrating the differential, $s = \int ds$.

Intuition: We can approximate the length of a curve with a polygonal path of line segments of the form

$$\Delta s_i = \sqrt{(\Delta x)^2 + (\Delta y_i)^2}.$$

By the mean value theorem, there exists a x_i^* in the subinterval of length Δx such that $\Delta y_i = f'(x_i^*)\Delta x$, so the approximation can be written as

$$\Delta s_i = \sqrt{(\Delta x)^2 + (f'(x_i^*)\Delta x)^2} = \sqrt{1 + (f'(x_i^*))^2}\Delta x. \tag{1}$$

If we partition $[a, b]$ into n uniform subintervals and approximate the area with a polygonal path of line segments of the form (1), taking the limit as $n \to \infty$ implies

$$\text{Arc Length} = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + (f'(x_i^*))^2}\Delta x = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

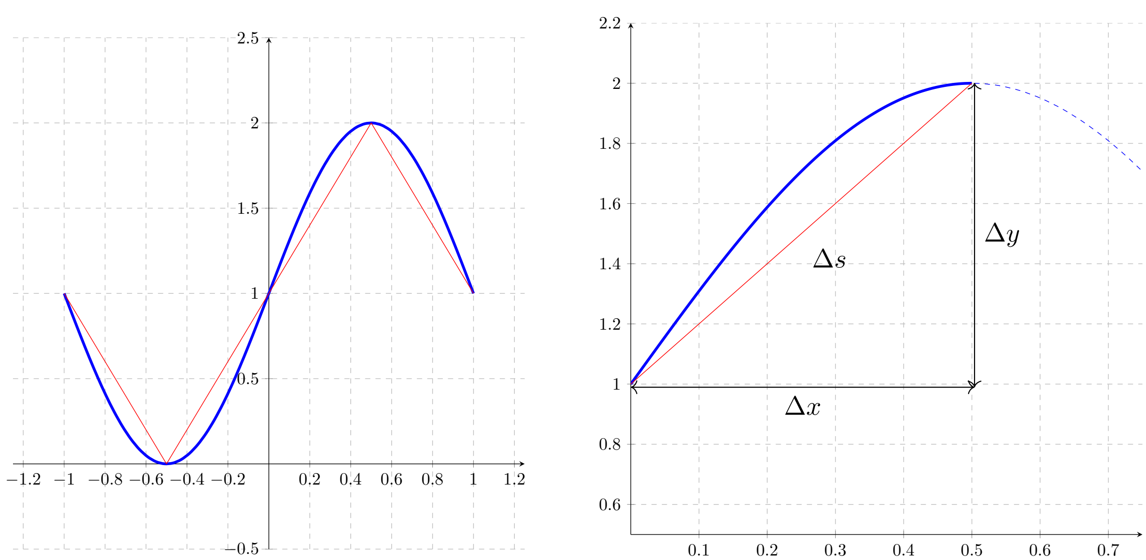


Figure: The length of the line segment can be recovered using the Pythagorean theorem. The length of each approximating line segment is given by

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2.$$

1.2 Surface Area

Formula: If $f'(x)$ is continuous on $[a, b]$, then the *surface area of a solid of revolution* obtained by rotating the curve $y = f(x)$

1. Around the y-axis on the interval $[a, b]$ is given by (provided that $x \geq 0$)

$$S = \int_a^b 2\pi x \sqrt{1 + (f'(x))^2}\, dx.$$

2. Around the x-axis on the interval $[a, b]$ is given by (provided that $y = f(x) \geq 0$)

$$S = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2}\, dx.$$

Remark. We can remember this formula using the differential notation

$$S = \int 2\pi x\, ds \quad (y\text{-axis rotation}) \qquad \text{or} \qquad S = \int 2\pi y\, ds \quad (x\text{-axis rotation}).$$

This surface area is recovered by integrating the circumference of a circle with respect to the arc length.

Intuition: If the surface it obtained by rotating about the y-axis, then we can approximate the surface area with a "trapezoidal" band (also called the *frustrum* of a cone) of the form

$$A_i = 2\pi \bar{x}_i \Delta s_i \approx 2\pi x_i^* \sqrt{1 + (f'(x_i^*))^2} \Delta x, \tag{2}$$

where $\bar{x}_i$ is a the midpoint of the subinterval and x_i^* is a point in a subinterval of length Δx. If we partition $[a, b]$ into n uniform subintervals and approximate the area with a polygonal path of line segments of the form (2), taking the limit as $n \to \infty$ implies

$$\text{Surface Area} = \lim_{n\to\infty} \sum_{i=1}^n 2\pi x_i^* \sqrt{1 + (f'(x_i^*))^2} \Delta x = \int_a^b 2\pi x \sqrt{1 + (f'(x))^2}\, dx.$$

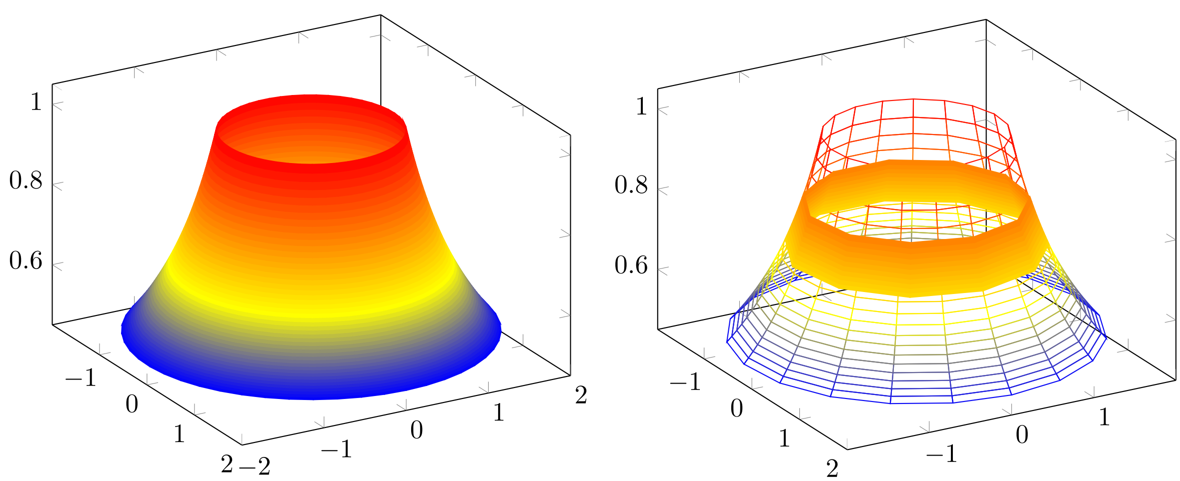


Figure: The area of the horizontal band can be estimated with a rectangle. The height of the rectangle is estimated by the arc length Δs, and the width of the rectangle is the circumference of a circle with radius x

$$\text{Area} = 2\pi x \times \Delta s.$$

Remark. If the surface it obtained by rotating about the x-axis, then the only modification is the radius of the circle is $y = f(x)$ instead.

1.3 Example Problems

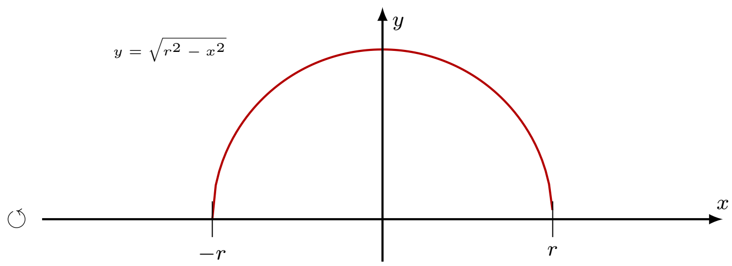
1.3.1 Arc Length

Strategy:

1. *(Optional) Draw the Curve:* Draw the curve in the (x, y) plane.
2. *Set up the definite integral:* Use the formula for the arc length treating the curve as a function of y or x (depending on which results in the simpler integral).
3. Compute the integral.

Problem 1.1. ($\star\star$) Show that the circumference of a circle with radius r is $2\pi r$.

Solution 1.1. By symmetry, it suffices to compute the arc length of the semi-circle $y = \sqrt{r^2 - x^2}$ on the domain $[-r, r]$ and multiply our final answer by 2.



Finding the Integral: Since $\frac{dy}{dx} = \frac{-x}{\sqrt{r^2-x^2}}$, the arc length of the semicircle is given by

$$\int_{-r}^{r} \sqrt{1 + \Big(\frac{dy}{dx}\Big)^2}\, dx = \int_{-r}^{r} \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx = \int_{-r}^{r} \frac{r}{\sqrt{r^2 - x^2}}\, dx.$$

Computing the Integral: The integrand looks like the derivative of the $\sin^{-1}(x)$, but we need to do some algebraic manipulation first. We multiply the numerator and denominator by r^{-1} to conclude

$$\int_{-r}^{r} \frac{r}{\sqrt{r^2 - x^2}}\, dx = \int_{-r}^{r} \frac{1}{\sqrt{1 - (\frac{x}{r})^2}}\, dx.$$

We will use the change of variables $u = \frac{x}{r}$,

$$\frac{du}{dx} = \frac{1}{r}, \Rightarrow r\,du = dx \qquad x = -r \to u = -1, \qquad x = r \to u = 1.$$

Under this change of variable, we have

$$\int_{-r}^{r} \frac{1}{\sqrt{1 - (\frac{x}{r})^2}}\, dx = \int_{-1}^{1} \frac{r}{\sqrt{1 - u^2}}\, du = r \sin^{-1}(u)\Big|_{u=-1}^{u=1} = r\Big(\frac{\pi}{2} + \frac{\pi}{2}\Big) = \pi r.$$

Therefore, the circumference of a semi-circle is πr and the circumference of the circle is $2\pi r$.

Remark: In general, arc length integrals are quite hard to compute because of the square root term in the integrand. We will learn more tools later such as trigonometric substitution that will allow us to solve much more difficult problems.

Problem 1.2. ($\star\star$) Find the arc length of the curve $f(x) = \frac{x^2}{8} - \ln(x)$ on the interval $[1, 2]$.

Solution 1.2. We have to use the arc length formula in terms of dx.

Finding the Integral: Since $\frac{dy}{dx} = \frac{x}{4} - \frac{1}{x}$, the formula for the arc length implies that

$$S = \int_1^2 \sqrt{1 + \Big(\frac{dy}{dx}\Big)^2}\, dx = \int_1^2 \sqrt{1 + \Big(\frac{x}{4} - \frac{1}{x}\Big)^2}\, dx = \int_1^2 \sqrt{\frac{x^2}{16} + \frac{1}{2} + \frac{1}{x^2}}\, dx.$$

Computing the Integral: Since $\frac{x^2}{16} + \frac{1}{2} + \frac{1}{x^2} = (\frac{x}{4} + \frac{1}{x})^2$,

$$\int_1^2 \sqrt{\frac{x^2}{16} + \frac{1}{2} + \frac{1}{x^2}}\, dx = \int_1^2 \frac{x}{4} + \frac{1}{x}\, dx = \frac{x^2}{8} + \ln(x)\Big|_{x=1}^{x=2} = \frac{3}{8} + \ln(2).$$

Problem 1.3. ($\star\star$) Find the length of the curve $x = \ln(\sec(y))$ on the domain $y \in [0, \frac{\pi}{4}]$.

Solution 1.3. We have to use the arc length formula in terms of dy.

Finding the Integral: Since $\frac{dx}{dy} = \frac{\sec(y)\tan(y)}{\sec(y)} = \tan(y)$, the arc length of the curve is given by

$$\int_0^{\frac{\pi}{4}} \sqrt{1 + \Big(\frac{dx}{dy}\Big)^2}\, dy = \int_{-r}^{r} \sqrt{1 + \tan^2(y)}\, dy = \int_0^{\frac{\pi}{4}} |\sec(y)|\, dy = \int_0^{\frac{\pi}{4}} \sec(y)\, dy$$

if we use the trigonometric identity $1 + \tan^2(\theta) = \sec^2\theta$.

Computing the Integral: This is a standard integral (see Week 10 Problem 2.6),

$$\int_0^{\frac{\pi}{4}} \sec(y)\, dy = \ln|\sec(y) + \tan(y)|\Big|_{y=0}^{y=\frac{\pi}{4}} = \ln(\sqrt{2} + 1).$$

1.3.2 Surface Area

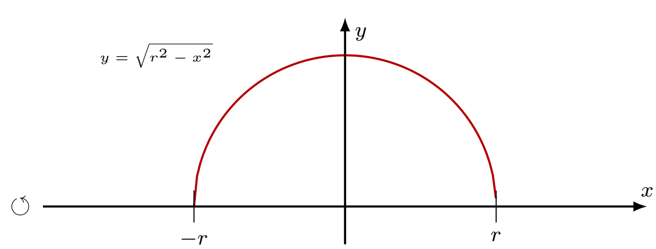
Strategy:

1. *(Optional) Draw the Projected Curve:* Draw projection of the curve onto the (x, y) plane.
2. *Set up the definite integral:* Find a formula for the surface area by using the surface area formulas.
3. Compute the integral.

Problem 1.4. ($\star\star$) Show that the surface area of a sphere with radius r is $4\pi r^2$.

Solution 1.4. We compute the surface area in two ways:

Rotating around the x-axis The sphere is obtained by rotating the curve $y = \sqrt{r^2 - x^2}$ on the domain $[-r, r]$ and around the x-axis.



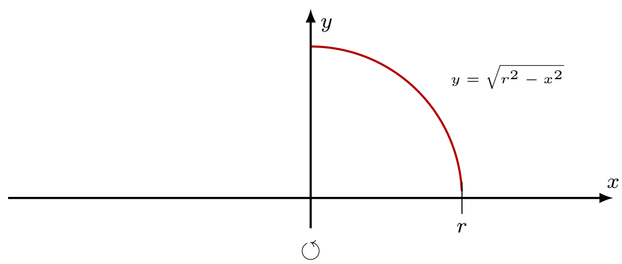
Finding the Integral: Since $\frac{dy}{dx} = \frac{-x}{\sqrt{r^2-x^2}}$, the surface area of the sphere is given by

$$\int_{-r}^{r} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_{-r}^{r} 2\pi\sqrt{r^2-x^2}\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx = 2\pi r\int_{-r}^{r} 1\,dx.$$

Computing the Integral: This integral is easy to compute,

$$2\pi r\int_{-r}^{r} 1\,dx = 2\pi r x\Big|_{x=-r}^{x=r} = 4\pi r^2.$$

Rotating around the y-axis By symmetry, it suffices to compute the surface area of the half sphere is obtained by rotating the curve $y = \sqrt{r^2-x^2}$ on the domain $[0,r]$ and around the y-axis and multiplying our final answer by 2.



Finding the Integral: Since $\frac{dy}{dx} = \frac{-x}{\sqrt{r^2-x^2}}$, the surface area of the sphere is given by

$$\int_{0}^{r} 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_{0}^{r} 2\pi x\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx = 2\pi r\int_{0}^{r} \frac{x}{\sqrt{r^2-x^2}}\,dx.$$

Computing the Integral: We will use the change of variables $u = r^2 - x^2$,

$$\frac{du}{dx} = -2x \Rightarrow \frac{1}{-2}du = xdx \qquad x = 0 \to u = r^2, \qquad x = r \to u = 0.$$

Under this change of variables, we have

$$2\pi r\int_{0}^{r} \frac{x}{\sqrt{r^2-x^2}}\,dx = -\pi r\int_{r^2}^{0} \frac{1}{\sqrt{u}}\,du = -2\pi r\sqrt{u}\Big|_{u=r^2}^{u=0} = -2\pi r\left(0-\sqrt{r^2}\right) = 2\pi r^2.$$

Therefore, the surface area of a hemisphere is $2\pi r^2$ and the surface area of the sphere is $4\pi r^2$.

2 Applications to Physics

2.1 Work

Definition 1. Recall that the *work* to move an object from the point $x = a$ to $x = b$ with variable force $F(x)$ is given by

$$W = \int_a^b F(x)\,dx.$$

Example 1. There are several physical examples where the formula for the force $F(x)$ is well known,

1. *Newton's Second Law of Motion:* Let m be the mass of an object and let a be the acceleration of the object, then the force required to move the object is
$$F(x) = ma.$$
2. *Hooke's Law:* Let $k > 0$ be the spring constant. If x is the distance from the spring's equilibrium point, then the force required to maintain the spring x units is
$$F(x) = kx.$$
3. *Gravitational Force:* Let m_1 and m_2 be the masses of two objects separated at a distance x. The gravitational force of attraction (with gravitational constant G) is
$$F(x) = \frac{Gm_1m_2}{x^2}.$$

2.2 Work Required to Move Fluids

2.2.1 Filling a Tank

Formula: The work to fill a tank with cross sectional area $A(y)$ with starting fluid height $y = a$ to ending height $y = b$ with a fluid of density ρ is given by

$$W = \int_a^b \rho g y A(y)\,dy \qquad \text{where } g \text{ is the acceleration of gravity.}$$

2.2.2 Emptying a Tank

Formula: The work to empty a tank with cross sectional area $A(y)$ with starting fluid height $y = a$ to ending height $y = b$ with a fluid of density ρ into container of height $y = h$ is given by

$$W = \int_a^b \rho g (h-y) A(y)\,dy \qquad \text{where } g \text{ is the acceleration of gravity.}$$

Intuition: The work to move the fluid is approximated by the sum of work required small shells of fluid. The work to move a small shell of fluid is modeled using Newton's law:

$$\text{Work} = \text{force} \times \text{distance} = \text{mass} \times \text{acceleration} \times \text{distance}.$$

The mass of a shell of fluid with cross-sectional area $A(y)$ is given by *density* $\times$ *volume*:

$$\text{mass} = \rho A(y)\Delta y. \qquad \text{(see Week 11 Section 2.1)}$$

To fill a tank, we need to move this shell of fluid a distance of height y, so

$$\text{Work} = \lim_{n\to\infty} \sum_{n=1}^{\infty} \rho A(y)\Delta y \times g \times y = \int_a^b \rho g y A(y)\,dy.$$

To empty a tank, we need to move this shell of fluid a distance of height $h - y$, so

$$\text{Work} = \lim_{n\to\infty} \sum_{n=1}^{\infty} \rho A(y)\Delta y \times g \times (h-y) = \int_a^b \rho g (h-y) A(y)\,dy.$$

2.3 Example Problems

Problem 2.1. ($\star$) A force of 40 N is required to hold a spring that has been stretched from a natural length of 0.1 m to a length of 0.15 m. How much work is required to stretch the spring from 0.15 m to 0.2 m?

Solution 2.1. We need to use Hooke's Law.

Finding the Integral: The work required to stretch the spring is

$$W = \int_{0.15}^{0.2} F(x)\,dx = \int_{0.15}^{0.2} k(x-0.1)\,dx,$$

since the equilibrium point of the spring is $x = 0.1$. To find the spring constant k, notice that

$$40 = k(0.15-0.1) \implies k = \frac{40}{0.05} = 800.$$

Compute the Integral: This integral is easy to compute,

$$W = \int_{0.15}^{0.2} 800(x-0.1)\,dx = 400x^2 - 80x\Big|_{x=0.15}^{x=0.2} = 3 \text{ (joules)}.$$

Problem 2.2. ($\star\star$) Consider a circular conical tank with height 10 m and radius 4 m. Suppose that the water level is currently 8 m high. Given that the density of water is $\rho = 1000$ kg/m^3, find the work required to empty the tank.

Solution 2.2. We need to use the formula to empty a tank.

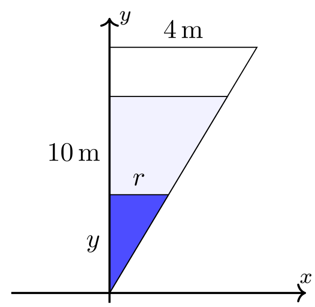
Finding the Integral: The work empty the fluid out of a tank 10m high is

$$W = \int_0^8 \rho g(10-y)A(y)\,dy.$$

Using similar triangles, the cross sectional area is a disc with radius

$$\frac{4}{10} = \frac{r}{y} \implies r = \frac{2}{5}y.$$

Therefore, the cross sectional area is $A(y) = \pi r^2 = \pi \frac{4}{25} y^2$.



Compute the Integral: This integral is easy to compute. Using the fact that $g = 9.8\,\text{m/s}^2$,

$$\begin{aligned}
\int_0^8 \rho g(10-y)A(y)\,dy &= \int_0^8 1000 \cdot 9.8 \cdot \pi \cdot \frac{4}{25}(10-y)y^2\,dy \\
&= 160 \cdot 9.8 \cdot \pi \Big(\frac{10}{3}y^3 - \frac{1}{4}y^4\Big)\Big|_{y=0}^{y=8} \\
&= 160 \cdot 9.8 \cdot \pi \Big(\frac{10}{3}8^3 - \frac{1}{4}8^4\Big) \\
&\approx 3.36 \times 10^6 \text{ (joules)}.
\end{aligned}$$