1 Using Integration to Find Areas

1.1 Areas Between Curves

Formula: The area between the two curves $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$ is given by

$$A = \int_a^b |f(x) - g(x)|\, dx.$$

Intuition: We can approximate the area with small rectangles of the form

$$A_i = |f(x_i^*) - g(x_i^*)|\Delta x, \tag{1}$$

where x_i^* is a point in a subinterval of length Δx. If we partition $[a, b]$ into n uniform subintervals and approximate the area with rectangles of the form (1), taking the limit as $n \to \infty$ implies

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} |f(x_i^*) - g(x_i^*)|\Delta x = \int_a^b |f(x) - g(x)|\, dx.$$

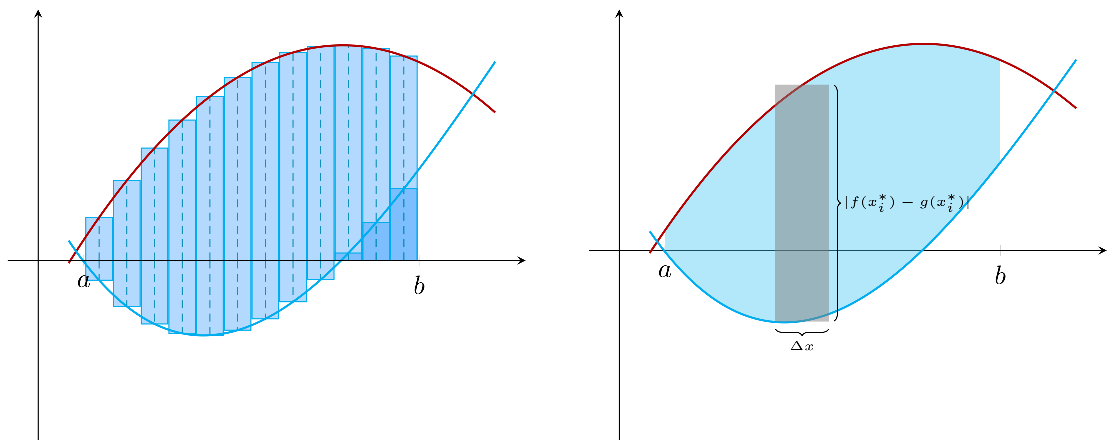


Figure: The height of the subrectangle is the distance between the two functions f and g. The area of each of the approximating rectangles is given by

$$\text{length} \times \text{width} = |f(x_i^*) - g(x_i^*)|\Delta x.$$

1.2 Total Distance Traveled

Net Distance: Let $v(t)$ be the velocity of a particle. The *net distance traveled* by the particle over the time interval $[a, b]$ is given by

$$\int_a^b v(t)\, dt.$$

The *average velocity* is given by

$$\frac{1}{b-a}\int_a^b v(t)\, dt.$$

Total Distance: Let $v(t)$ be the velocity of a particle. The *total distance traveled* by the particle over the time interval $[a, b]$ is given by

$$\int_a^b |v(t)|\, dt.$$

The *average speed* is given by

$$\frac{1}{b-a}\int_a^b |v(t)|\, dt.$$

1.3 Example Problems

1.3.1 Distance/Displacement Problems

Problem 1. ($\star\star$) Let $v(t) = 1 - \ln(1+t)$ be the speed of a particle for $0 \le t \le 5$.

1. Find the average velocity of the particle.
2. Find the average speed of the particle.

Solution 1.

Part (a) *Average Velocity:* The net distance is traveled is given by,

$$\begin{aligned}\int_0^5 (1-\ln(1+t))\,dt &= t - t\ln(1+t)\Big|_{t=0}^{t=5} + \int_0^5 \frac{t}{1+t}\,dt && \text{integration by parts}\\ &= t - t\ln(1+t)\Big|_{t=0}^{t=5} + \int_0^5 1 - \frac{1}{1+t}\,dt && \text{long division}\\ &= t - t\ln(1+t) + t - \ln(1+t)\Big|_{t=0}^{t=5}\\ &= 10 - 6\ln(6) \approx -0.7506.\end{aligned}$$

The average velocity is therefore,

$$\frac{1}{5}\int_0^5 (1-\ln(1+t))\,dt = \frac{1}{5}\Big(10 - 6\ln(6)\Big) \approx -0.15.$$

Part (b) *Average Speed:* The total distance traveled is given by

$$\int_0^5 |1-\ln(1+t)|\,dt.$$

We first classify the signs of $f(t) = 1 - \ln(1+t)$. The roots are given by

$$1 - \ln(1+t) = 0 \Rightarrow 1 + t = e \Rightarrow t = e - 1.$$

The signs are also given by

Therefore, the integral is given by

$$\begin{aligned}\int_0^5 |1-\ln(1+t)|\,dt &= \int_0^{e-1} (1-\ln(1+t))\,dt - \int_{e-1}^5 (1-\ln(1+t))\,dt && \text{definition of } |\cdot|\\ &= 2t - (1+t)\ln(1+t)\Big|_{t=0}^{t=e-1} - (2t - (1+t)\ln(1+t))\Big|_{t=e-1}^{t=5} && \text{same steps as Part(a)}\\ &= 2(e-1) - e\ln(e) - (10 - 6\ln(6) - 2(e-1) + e\ln(e))\\ &= -14 + 2e + 6\ln(6) \approx 2.1871.\end{aligned}$$

The average speed is therefore,

$$\frac{1}{5}\int_0^5 |1-\ln(1+t)|\,dt = \frac{1}{5}\Big(-14 + 2e + 6\ln(6)\Big) \approx 0.437.$$

1.3.2 Areas Between Curves

Strategy: The areas between curves can be computed without drawing a picture.

1. *(Optional) Draw the Curves:* Draw the curves on the (x, y) plane.
2. *Set up the definite integral:* Find the functions that represents the curves and the domain of integration. It may be useful to treat our curves as a function of y instead of x in some examples.
3. *Write the absolute value as a piecewise function:* Find the regions where $f(x) - g(x) > 0$ and $f(x) - g(x) < 0$ and split the region of integration into the different regions.
4. Compute the integrals.

Problem 1. ($\star$) Find the area of the region bounded by the curves $y = x^2$ and $y = \sqrt{x}$.

Solution 1.

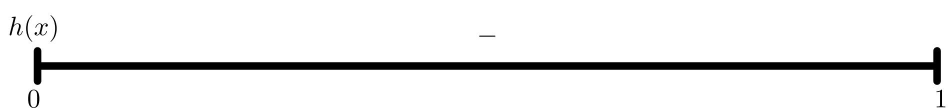
Finding the Integral: We first start by expressing the area as a definite integral. The first curve is given by $y = x^2$ and the second curve is given by $y = \sqrt{x}$. The curves intersect when

$$x^2 = \sqrt{x} \Rightarrow x^4 = x \Rightarrow x(x^3 - 1) = 0 \Rightarrow x = 0, 1.$$

The region of integration is given by the smallest and the largest of these values, so the area by

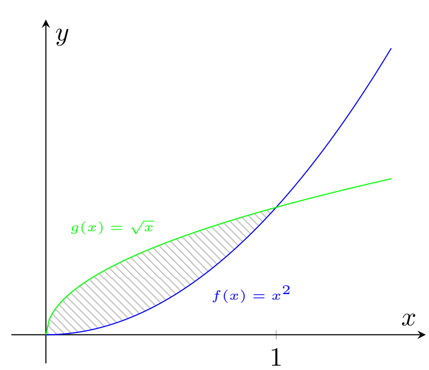
$$\int_0^1 |x^2 - \sqrt{x}|\, dx.$$

Compute the Integral: We first classify the signs of $h(x) = x^2 - \sqrt{x}$. From the first part, we found that the roots are given by $0, 1$ so the signs are given by



Therefore, the area is given by

$$\int_0^1 |x^2 - \sqrt{x}|\, dx = -\int_0^1 (x^2 - \sqrt{x})\, dx = -\frac{x^3}{3} + \frac{2}{3}x^{3/2}\Big|_{x=0}^{x=1} = \frac{1}{3}.$$



Problem 2. ($\star\star$) Find the area of the region bounded by the curves $y^2 + x = 1$ and $y^2 - x = 1$.

Solution 2. This problem is must easier to do if we treat x as a function of y.

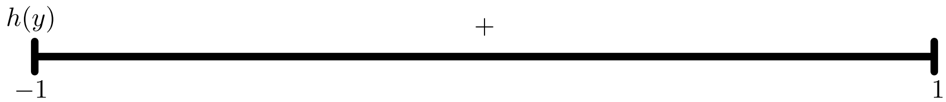
Finding the Integral: Our functions are given by $x = 1 - y^2$ and $x = y^2 - 1$. The curves intersect when

$$1 - y^2 = y^2 - 1 \Rightarrow 2y^2 - 2 = 0 \Rightarrow y^2 - 1 = 0 \Rightarrow y = \pm 1.$$

Therefore, the integral is given by

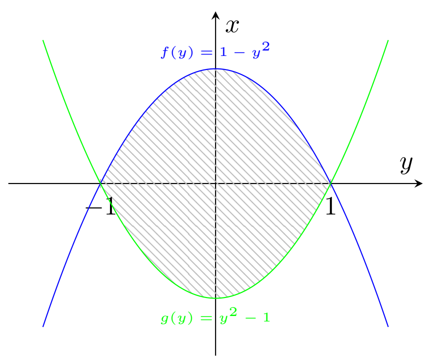
$$\int_{-1}^{1} |1 - y^2 - (y^2 - 1)| \, dy = \int_{-1}^{1} |2 - 2y^2| \, dy.$$

Compute the Integral: We first classify the signs of $h(y) = 2 - 2y^2$. From the first part, we found that the roots are given by $-1, 1$ so the signs are given by



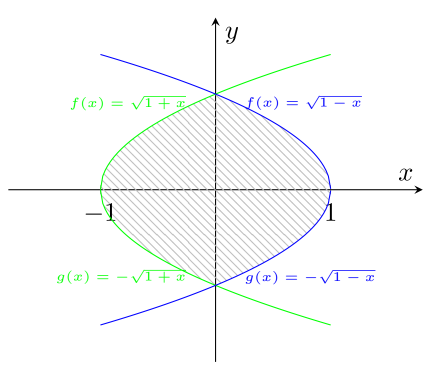
Therefore, the area is given by

$$\int_{-1}^{1} |2 - 2y^2| \, dy = \int_{-1}^{1} 2 - 2y^2 \, dy = 2y - \frac{2}{3}y^3 \Big|_{y=-1}^{y=1} = 4 - \frac{4}{3} = \frac{8}{3}.$$



Remark: If we integrated with respect to x, then we would have computed

$$\int_{-1}^{0} \sqrt{1+x} + \sqrt{1+x} \, dx + \int_{0}^{1} \sqrt{1-x} + \sqrt{1-x} \, dx = \frac{4}{3}(1+x)^{3/2} \Big|_{x=-1}^{x=0} - \frac{4}{3}(1-x)^{3/2} \Big|_{x=0}^{x=1} = \frac{8}{3}.$$



2 Using Integration to Find Volumes

2.1 Volumes Using Cross-Sectional Area

Formula: The *volume of a solid* with cross-sectional areas $A(x)$ perpendicular to the x-axis from $x = a$ to $x = b$ is

$$V = \int_a^b A(x)\,dx.$$

Intuition: We can approximate the volume with small cylinders of the form

$$V_i = A(x_i^*)\Delta x, \tag{2}$$

where x_i^* is a point in a subinterval of length Δx. If we partition $[a, b]$ into n uniform subintervals and approximate the area with cylinders of the form (2), taking the limit as $n \to \infty$ implies

$$\text{Volume} = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\,dx.$$

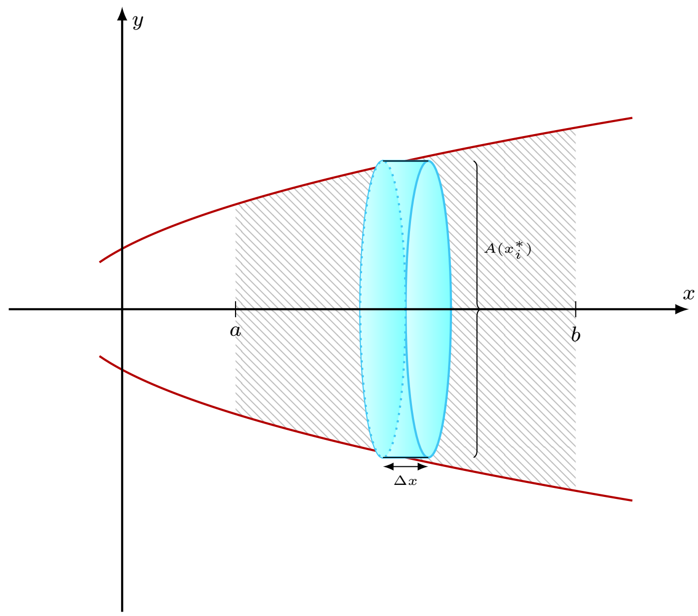


Figure: The base area of the cylinder is $A(x_i^*)$ and the height of the cylinder is Δx. The area of each of the approximating cylinders is given by

$$\text{base area} \times \text{height} = A(x_i^*)\Delta x.$$

2.2 Volumes Using Washers (Rotation around a Horizontal Axis)

Formula: The *volume of the solid of revolution* rotated about a *horizontal* axis with outer radius $R(x)$ and inner radius $r(x)$ from $x = a$ to $x = b$ is

$$V = \int_a^b \left(\pi R(x)^2 - \pi r(x)^2\right) dx.$$

Intuition: This formula is a special case of the volumes using cross-sectional area when the cross-sectional area of the solid is a annulus with inner $r(x)$ and outer radius $R(x)$. The cross sectional area is given explicitly by

$$A(x) = \pi R(x)^2 - \pi r(x)^2.$$

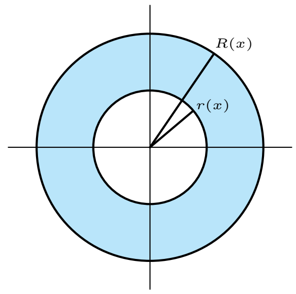


Figure: The cross-sectional area of a solid generated by rotation around a horizontal axis is given by

$$\text{Area Outer Circle} - \text{Area Inner Circle} = \pi R(x)^2 - \pi r(x)^2.$$

2.3 Volumes Using Shells (Rotation around a Vertical Axis)

Formula: The *volume of the solid of revolution* rotated about a *vertical* axis with upper height $H(x)$ and lower height $h(x)$ from $x = a$ to $x = b$ at a (positive) distance $r(x)$ from the axis of revolution is

$$V = \int_a^b 2\pi r(x)\big(H(x) - h(x)\big)\,dx.$$

Intuition: We can approximate the volume with small cylinders of the form

$$V_i = 2\pi r(x_i^*)\big(H(x_i^*) - h(x_i^*)\big)\Delta x = 2\pi r(x_i^*)\big(H(x_i^*) - h(x_i^*)\big)\Delta x, \tag{3}$$

where x_i^* is a point in a subinterval of length Δx. If we partition $[a, b]$ into n uniform subintervals and approximate the area with cylinders of the form (3), taking the limit as $n \to \infty$ implies

$$\text{Volume} = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi r(x_i^*)\big(H(x_i^*) - h(x_i^*)\big)\Delta x = \int_a^b 2\pi r(x)\big(H(x) - h(x)\big)\,dx.$$

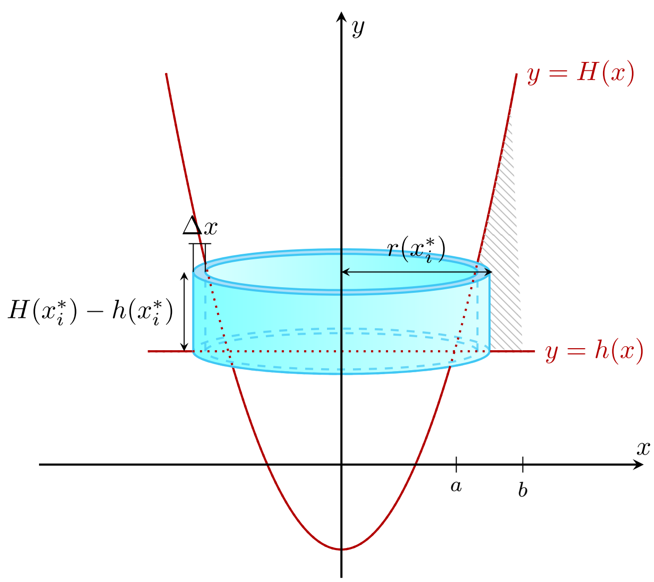


Figure: The length of the cylindrical shell is given by the radius of a circle, length $= 2\pi r(x_i^*)$. The area of each of the approximating cylindrical shells is given by

$$\text{length} \times \text{height} \times \text{width} = 2\pi r(x_i^*)\big(H(x_i^*) - h(x_i^*)\big)\Delta x.$$

Remark: If the rotation is about the y-axis, and $0 \le a < b$ (the region is to the right of the axis of rotation), then the radius $r(x) = x$ and the formula is $V = \int_a^b 2\pi x\big(H(x) - h(x)\big)\,dx$.

2.4 Example Problems:

Strategy:

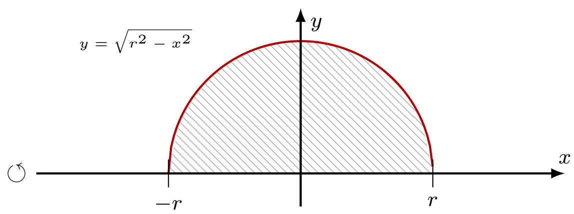
1. *(Optional) Draw the Projected Area:* Draw the area of the curve projected onto the (x, y) plane.
2. *Set up the definite integral:* Find a formula for volume using either the cross sectional area or cylindrical shells. Choose the representation that will result in a simpler integral.
3. Compute the integral.

Remark: Sometimes it might be more convenient to integrate with respect to y instead of x. All the formulas in this section can be easily modified to by interchanging the axes (replace the x variable with a y variable and treat all functions as functions of y instead of x).

Problem 1. ($\star$) Compute the volume of a ball with radius r.

Solution 1. The area can be computed using either washers or cylindrical shells.

Washers: The ball is generated by rotating the area under the curve of $y = \sqrt{r^2 - x^2}$ on the interval $[-r, r]$ around the x-axis.



Finding the Integral: The cross-sectional area is a circle with radius $\sqrt{r^2 - x^2}$. Therefore, the cross-sectional area is given by

$$A(x) = \pi(r^2 - x^2).$$

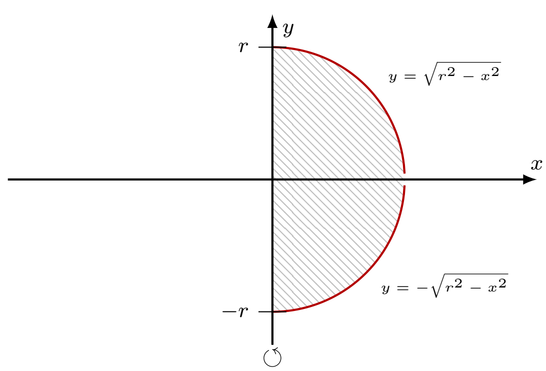
Using the volume formula for washers, the volume integral is

$$\int_{-r}^{r} \pi(r^2 - x^2)\, dx.$$

Computing the Integral: The integrand is even, so

$$\int_{-r}^{r} \pi(r^2 - x^2)\, dx = 2\int_{0}^{r} \pi(r^2 - x^2)\, dx = 2\pi r^2 x - 2\pi \frac{x^3}{3}\Big|_{x=0}^{x=r} = \frac{4}{3}\pi r^3.$$

Cylindrical Shells: The ball is generated by rotating the area bounded by the curves $y = \sqrt{r^2 - x^2}$ and $y = -\sqrt{r^2 - x^2}$ on the interval $[0, r]$ around the y-axis.



Finding the Integral: The radius of the cylinder is given by $r(x) = x$ and the height of the cylinder is given by

$$H(x) - h(x) = \sqrt{r^2 - x^2} - (-\sqrt{r^2 - x^2}) = 2\sqrt{r^2 - x^2}.$$

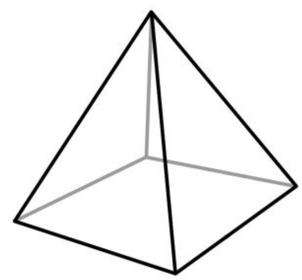
Using the volume formula for cylindrical shells, the volume integral is

$$\int_0^r 2\pi x(2\sqrt{r^2 - x^2})\,dx = \int_0^r 4\pi x\sqrt{r^2 - x^2}\,dx.$$

Computing the Integral: Using the substitution $u = r^2 - x^2$, we have

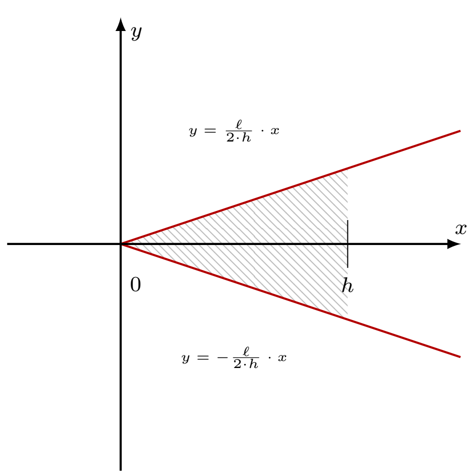
$$\int_0^r 4\pi x\sqrt{r^2 - x^2}\,dx = -2\pi \cdot \frac{2}{3}(r^2 - x^2)^{3/2}\Big|_{x=0}^{x=r} = \frac{4}{3}\pi r^3.$$

Problem 2. ($\star$) Compute the volume of square pyramid with base length ℓ and height h.



Solution 2. The area can be computed using cross sectional area.

Cross-Sectional Area: We will orient the pyramid along the x-axis with vertex at the origin. The upper edge of the pyramid must pass through the point $(h, \ell/2)$, so the height is given by $y = \frac{\ell}{2h} \cdot x$. Similarly the height of the lower edge of the pyramid is given by $y = -\frac{\ell}{2h} \cdot x$.



Finding the Integral: The cross-sectional area is a square with side length $2 \cdot \frac{\ell}{2h}x$. Therefore, the cross sectional-area is given by

$$A(x) = \left(2 \cdot \frac{\ell}{2h}x\right)^2 = \frac{\ell^2}{h^2}x^2.$$

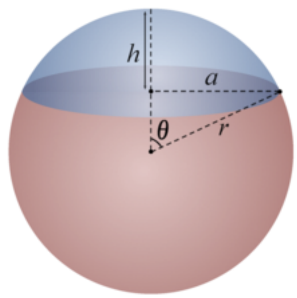
Using the volume formula for cross-sections, the volume integral is

$$\int_0^h \frac{\ell^2}{h^2} x^2 \, dx.$$

Computing the Integral: This integral is easy to compute,

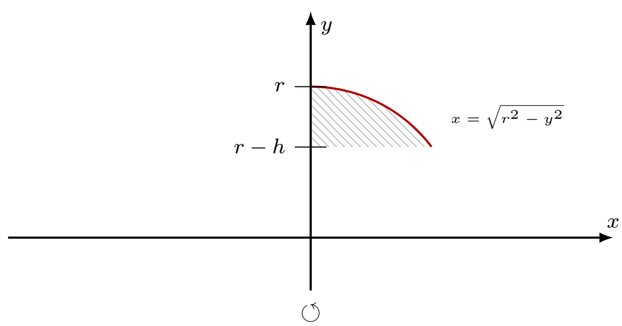
$$\int_0^h \frac{\ell^2}{h^2} x^2 \, dx = \frac{\ell^2}{3h^2} x^3 \Big|_{x=0}^{x=h} = \frac{\ell^2}{3h^2} h^3 = \frac{1}{3} \ell^2 h.$$

Problem 3. ($\star\star$) Compute the volume of a spherical cap with height $h < r$ from a ball radius r.



Solution 3. The area can be computed using either washers or cylindrical shells. The method with cylindrical shells is a bit harder in this case.

Washers: The spherical cap is generated by rotating the area under the curve of $x = \sqrt{r^2 - y^2}$ on the interval $[r-h, r]$ around the y-axis.



Finding the Integral: The cross-sectional area is a circle with radius $\sqrt{r^2 - y^2}$. Therefore, the cross-sectional area is given by

$$A(y) = \pi(r^2 - y^2).$$

Using the volume formula for washers, the volume integral is

$$\int_{r-h}^{r} \pi(r^2 - y^2) \, dy.$$

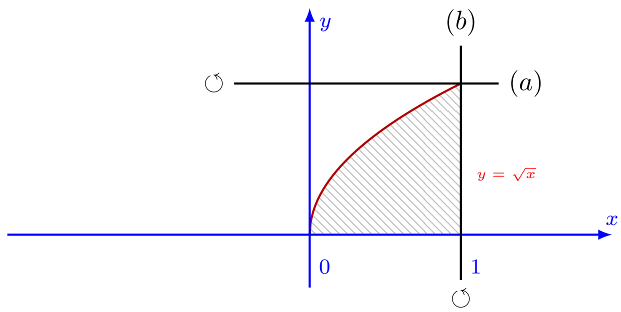
Computing the Integral: This integral is easy to compute,

$$\int_{r-h}^{r} \pi(r^2 - y^2) \, dy = \pi r^2 y - \frac{\pi}{3} y^3 \Big|_{y=r-h}^{y=r} = \frac{1}{3} \pi h^2 (3r - h).$$

Problem 4. ($\star\star$) Find the volume of the region bounded by $y = \sqrt{x}$, $y = 0$ and $x = 1$ rotated about

(a) the line $y = 1$

(b) the line $x = 1$.



Solution 4.

(a) We will compute the first volume using a washer.

Finding the Integral: To reduce this problem to the case of a rotation around the x-axis, we do a change of variable $\tilde{y} = y - 1$. In this case, we have the axis of rotation is the axis $\tilde{y} = 0$.

The cross-sectional area is a circle with outer radius $\tilde{y} = 0 - 1 = -1$ and inner radius $\tilde{y} = \sqrt{x} - 1$ (the signs do not matter because we will be squaring the radius to get the area). Therefore, the cross-sectional area is given by

$$A(x) = \pi((-1)^2 - (\sqrt{x} - 1)^2) = 2\pi\sqrt{x} - \pi x.$$

Using the volume formula for washers, the volume integral is

$$\int_0^1 2\pi\sqrt{x} - \pi x \, dx.$$

Computing the Integral: This integral is easy to compute,

$$\int_0^1 2\pi\sqrt{x} - \pi x \, dx = 2\pi\frac{2}{3}x^{3/2} - \pi\frac{x^2}{2}\Big|_{x=0}^{x=1} = \frac{5\pi}{6}.$$

Remark: If we integrated with respect to y using the cylindrical shell general formula, we would get

$$\int_0^1 2\pi(1-y)(1-y^2)\, dy = 2\pi\int_0^1 1 - y - y^2 + y^3 \, dy = 2\pi\Big(y - \frac{y^2}{2} - \frac{y^3}{3} + \frac{y^4}{4}\Big)\Big|_{y=0}^{y=1} = \frac{5\pi}{6}.$$

(b) We will compute the second volume using a cylindrical shell.

Finding the Integral: To reduce this problem to the case of a rotation around the y-axis, we do a change of variable $\tilde{x} = x - 1$. In this case, we have the axis of rotation is the axis $\tilde{x} = 0$. However, our region lies to the left of the axis of rotation, so we have to modify the volume formula slightly.

The radius of the cylinder is $-\tilde{x}$ (since the surface is to the left of the axis of rotation) and the height of the cylindrical shell is given by $y = \sqrt{x} = \sqrt{\tilde{x} + 1}$. The region of integration changes from $x \in [0, 1]$ to $\tilde{x} \in [-1, 0]$ Using the volume formula for cylindrical shells, the volume integral is

$$\int_{-1}^0 2\pi(-\tilde{x})\sqrt{\tilde{x}+1}\, d\tilde{x} = -\int_{-1}^0 2\pi\tilde{x}\sqrt{\tilde{x}+1}\, d\tilde{x}.$$

Computing the Integral: This integral can be computed using integration by parts,

$$-\int_{-1}^{0} 2\pi\tilde{x}\sqrt{\tilde{x}+1}\,d\tilde{x} = -2\pi\Big(\frac{2}{3}\tilde{x}(\tilde{x}+1)^{3/2} - \frac{4}{15}(\tilde{x}+1)^{5/2}\Big)\Big|_{\tilde{x}=-1}^{\tilde{x}=0} = \frac{8\pi}{15}.$$

Remark: If we used the general cylindrical shell formula, we have $r(x) = 1 - x$ giving us the volume

$$\int_0^1 2\pi(1-x)\sqrt{x}\,dx = 2\pi \cdot \frac{2}{3}x^{3/2} - 2\pi \cdot \frac{2}{5}x^{5/2}\Big|_{x=0}^{x=1} = \frac{8\pi}{15}.$$

Remark: If we integrated with respect to y using the washer formula, we would get

$$\int_0^1 \pi(1-y^2)^2\,dy = \pi\int_0^1 1 - 2y^2 + y^4\,dy = \pi \cdot \Big(y - \frac{2}{3}y^3 + \frac{y^5}{5}\Big)\Big|_{y=0}^{y=1} = \frac{8\pi}{15}.$$