1 Techniques of Integration

1.1 Integration By Substitution (Change of Variables)

We can think of integration by substitution as the counterpart of the chain rule for differentiation. Suppose that $g(x)$ is a differentiable function and f is continuous on the range of g. Integration by substitution is given by the following formulas:

Indefinite Integral Version:

$$\int f(u)\,du = \int f(g(x))g'(x)\,dx \qquad \text{where } u = g(x).$$

Definite Integral Version:

$$\int_{g(a)}^{g(b)} f(u)\,du = \int_a^b f(g(x))g'(x)\,dx \qquad \text{where } u = g(x).$$

1.2 Integration By Parts

We can think of integration by substitution as the counterpart of the product rule for differentiation. Suppose that $u(x)$ and $v(x)$ are continuously differentiable functions. Integration by parts is given by the following formulas:

Indefinite Integral Version:

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx.$$

Definite Integral Version:

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_{x=a}^{x=b} - \int_a^b u'(x)v(x)\,dx.$$

1.3 Average Value of a Function

The average value of a function f on the interval $[a, b]$ is given by

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

We can interpret f_{ave} as the number such that half the area of the curve lies above it, and half the area lies below it. In other words, f_{ave} is the number such that

$$\int_a^b (f(x) - f_{ave})\,dx = 0.$$

Example: The average of $f(x) = x^2$ and $g(x) = x + 1$ on the interval $[-2, 2]$ is displayed below:

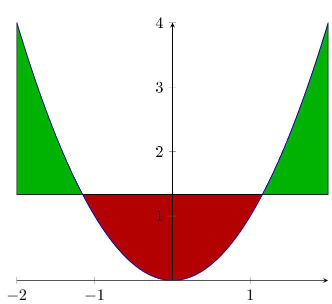

Figure 1: $f_{ave} = \frac{1}{4}\int_{-2}^{2} x^2\,dx = \frac{4}{3}$

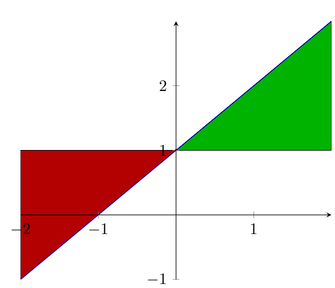

Figure 2: $g_{ave} = \frac{1}{4}\int_{-2}^{2} x + 1\,dx = 1$

1.4 More Properties of Integration

1. Integration of Odd Functions: If $f(x)$ is odd, then
$$\int_{-a}^{a} f(x)\,dx = 0.$$

2. Integration of Even Functions: If $f(x)$ is even, then
$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

3. The following theorem says that a continuous function attains its average value:

   Theorem 1 (The Mean Value Theorem for Integrals)**.** *If f is continuous on $[a,b]$, then there exists a number $c \in [a,b]$ such that $f(c) = f_{ave}$. That is, there exists a $c \in [a,b]$ such that*
$$\int_{a}^{b} f(x)\,dx = f(c)(b-a).$$

4. We can "move" the average integral inside of a function that is concave up:

   Theorem 2 (Jensen's Inequality)**.** *If $f(x)$ is a continuous function on $[a,b]$ and $g''(x) \geq 0$ for all x in the range of f, then $g(f_{ave}) \leq (g \circ f)_{ave}$. That is,*
$$g\left(\frac{1}{b-a}\int_{a}^{b} f(x)\,dx\right) \leq \frac{1}{b-a}\int_{a}^{b} g(f(x))\,dx.$$

1.5 Example Problems

1.5.1 Integration by Substitution

Strategy: The idea is to make the integral easier to compute by doing a change of variables.

1. Start by guessing what the appropriate change of variable $u = g(x)$ should be. Usually you choose u to be the function that is "inside" the function.
2. Differentiate both sides of $u = g(x)$ to conclude $du = g'(x)dx$. If we have a definite integral, use the fact that $x = a \to u = g(a)$ and $x = b \to u = g(b)$ to also change the bounds of integration.
3. Rewrite the integral by replacing all instances of x with the new variable and compute the integral or definite integral.
4. If you computed the indefinite integral, then make sure to write your final answer back in terms of the original variables.

Problem 1. ($\star$) Find the following indefinite integral
$$\int \tanh(x)\,dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$$

Solution 1.

Step 1: We will use the change of variables $u = e^x + e^{-x}$,

$$\frac{du}{dx} = e^x - e^{-x} \Rightarrow du = (e^x - e^{-x})\,dx.$$

Step 2: We can now evaluate the integral under this change of variables,

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \int \frac{du}{u} = \ln|u| + C$$
$$= \ln|e^x + e^{-x}| + C. \qquad u = e^x + e^{-x}$$

Since $e^x + e^{-x} > 0$, we can remove the absolute values if we wish giving the final answer

$$\int \tanh(x)\,dx = \ln(e^x + e^{-x}) + C.$$

Remark: We can use the fact $e^x + e^{-x} = 2\cosh(x)$ to conclude that

$$\ln(e^x + e^{-x}) + C = \ln(2\cosh(x)) + C = \ln(\cosh(x)) + \underbrace{\ln(2) + C}_{D} = \ln(\cosh(x)) + D.$$

This form of the indefinite integral may be easier to remember since it mirrors the fact that

$$\int \tan(x)\,dx = -\ln|\cos(x)| + C.$$

Problem 2. ($\star$) Evaluate the following integral

$$\int_0^1 x\sqrt{1-x^2}\,dx.$$

Solution 2.

Step 1: We will use the change of variables $u = 1 - x^2$,

$$\frac{du}{dx} = -2x \Rightarrow du = -2x\,dx \Rightarrow -\frac{1}{2}du = xdx, \qquad x = 0 \to u = 1, \qquad x = 1 \to u = 0.$$

Step 2: We can now evaluate the integral under this change of variables,

$$\int_0^1 x\sqrt{1-x^2}\,dx = -\frac{1}{2}\int_1^0 \sqrt{u}\,du = -\frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_{u=1}^{u=0} = \frac{1}{3}.$$

Remark: Instead of changing the bounds of integration, we can first find the indefinite integral,

$$\int x\sqrt{1-x^2}\,dx = -\frac{1}{2}(1-x^2)^{\frac{3}{2}},$$

then use the fundamental theorem of calculus to conclude

$$\int_0^1 x\sqrt{1-x^2}\,dx = -\frac{1}{2}(1-x^2)^{\frac{3}{2}}\Big|_{x=0}^{x=1} = \frac{1}{3}.$$

Problem 3. ($\star\star$) Find the following indefinite integral

$$\int \frac{1}{1+\sqrt{x}}\,dx.$$

Solution 3.

Step 1: We will use the change of variables $u = \sqrt{x}$. Then

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow 2\sqrt{x}du = dx \Rightarrow 2u\,du = dx.$$

Step 2: We can now evaluate the integral under this change of variables,

$$\int \frac{1}{1+\sqrt{x}}\,dx = \int \frac{2u}{1+u}\,du.$$

This integral is a bit tricky to compute, so we have to use algebra to simplify it first. Using long division to first simplify the integrand, we get

$$\begin{aligned}\int \frac{2u}{1+u}\,du = 2\int \frac{u}{1+u}\,du &= 2\int 1 - \frac{1}{1+u}\,du \\ &= 2u - 2\ln|1+u| + C \\ &= 2\sqrt{x} - 2\ln|1+\sqrt{x}| + C. \qquad u = \sqrt{x}.\end{aligned}$$

Problem 4. ($\star\star$) Find the following indefinite integral

$$\int \sec(x)\,dx.$$

Solution 4. We first do a trick by multiplying the numerator and denominator by $\sec(x) + \tan(x)$,

$$\int \sec(x)\,dx = \int \frac{\sec(x)(\sec(x)+\tan(x))}{\sec(x)+\tan(x)}\,dx = \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx.$$

Step 1: We will use the change of variables $u = \sec(x) + \tan(x)$,

$$\frac{du}{dx} = \sec(x)\tan(x) + \sec^2(x) \Rightarrow du = (\sec(x)\tan(x) + \sec^2(x))\,dx.$$

Step 2: We can now evaluate the integral under this change of variables,

$$\begin{aligned}\int \sec(x)\,dx = \int \frac{\sec^2(x)+\sec(x)\tan(x))}{\sec(x)+\tan(x)}\,dx &= \int \frac{1}{u}\,du \\ &= \ln|u| + C \\ &= \ln|\sec(x)+\tan(x)| + C. \qquad u = \sec(x)+\tan(x)\end{aligned}$$

Problem 5. ($\star\star$) Find the following indefinite integral

$$\int \operatorname{sech}(x)\,dx = \int \frac{2}{e^x + e^{-x}}\,dx.$$

Solution 5.

Step 1: We will use the change of variables $u = e^x$,

$$\frac{du}{dx} = e^x \Rightarrow dx = \frac{1}{e^x}\,du \Rightarrow dx = \frac{1}{u}\,du.$$

Step 2: We can now evaluate the integral under this change of variables,

$$\begin{aligned}\int \operatorname{sech}(x)\,dx = \int \frac{2}{e^x + e^{-x}}\,dx &= \int \frac{2}{u(u+u^{-1})}\,du \\ &= \int \frac{2}{u^2+1}\,du \\ &= 2\tan^{-1}(u) + C \\ &= 2\tan^{-1}(e^x) + C. \qquad u = e^x\end{aligned}$$

Alternative Solution: We first do a trick by multiplying the numerator and denominator by e^x,

$$\int \operatorname{sech}(x)\,dx = \int \frac{2}{e^x + e^{-x}}\,dx = \int \frac{2e^x}{e^{2x}+1}\,dx.$$

Step 1: We will use the change of variables $u = e^x$,

$$\frac{du}{dx} = e^x \Rightarrow du = e^x\,dx.$$

Step 2: We can now evaluate the integral under this change of variables,

$$\begin{aligned}\int \operatorname{sech}(x)\,dx = \int \frac{2e^x}{e^{2x}+1}\,dx &= \int \frac{2}{u^2+1}\,du \\ &= 2\tan^{-1}(u) + C \\ &= 2\tan^{-1}(e^x) + C. \qquad u = e^x\end{aligned}$$

1.5.2 Integration by Parts

We will introduce a method to bookkeep multiple integration by parts steps simultaneously. This is called the tabular method for integration by parts. You pick a term to differentiate and a term to integrate then repeat the operation until product of the terms in the last entry of the table is easy to integrate.

The integral can be recovered by multiplying diagonally across the rows of the table adding up all terms with alternating signs. The last term in the table is integrated across.

For example, the formula to integrate $\int u(x)v'''(x)\,dx$ by parts can be encoded by the table

$\pm$	D	I
$+$	u	v'''
$-$	u'	v''
$+$	u''	v'
$-\int$	u'''	v

which gives us the formula

$$\int u(x)v'''(x)\,dx = u(x)v''(x) - u'(x)v'(x) + u''(x)v(x) - \int u'''(x)v(x)\,dx.$$

1.6 Examples

Problem 1. ($\star$) Compute

$$\int xe^x\,dx.$$

Solution 1.

Step 1: Draw the table

$\pm$	D	I
$+$	x	e^x
$-$	1	e^x
$+\int$	0	e^x

Step 2: From the table, we have

$$\int xe^x\,dx = xe^x - e^x + C.$$

Problem 2. ($\star\star$) Compute

$$\int x^6e^x\,dx.$$

Solution 2.

Step 1: Draw the table

$\pm$	D	I
$+$	x^6	e^x
$-$	$6x^5$	e^x
$+$	$30x^4$	e^x
$-$	$120x^3$	e^x
$+$	$360x^2$	e^x
$-$	$720x$	e^x
$+$	720	e^x
$-\int$	0	e^x

Step 2: From the table, we have

$$\int x^6e^x\,dx = x^6e^x - 6x^5e^x + 30x^4e^x - 120x^2e^x - 720xe^x + 720e^x + C.$$

Problem 3. ($\star\star$) Compute

$$\int x^4 \sin x \, dx.$$

Solution 3.

Step 1: Draw the table

$\pm$	D	I
$+$	x^4	$\sin x$
$-$	$4x^3$	$-\cos x$
$+$	$12x^2$	$-\sin x$
$-$	$24x$	$\cos x$
$+$	24	$\sin x$
$-\int$	0	$-\cos x$

Step 2: From the table, we have

$$\int x^4 \sin x \, dx = -x^4 \cos x + 4x^3 \sin x + 12x^2 \cos x - 24x \sin x - 24 \cos x + C.$$

Problem 4. ($\star\star$) Compute

$$\int e^x \sin x \, dx.$$

Solution 4.

Step 1: Draw the table

$\pm$	D	I
$+$	$\sin x$	e^x
$-$	$\cos x$	e^x
$+\int$	$-\sin x$	e^x

Step 2: From the table, we have

$$\int e^x \sin x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx + D.$$

Moving all the $\int e^x \sin x \, dx$ to one side and simplifying, we can conclude

$$2 \int e^x \sin x \, dx = e^x \sin x - e^x \cos x + D \implies \int e^x \sin x \, dx = \frac{1}{2} e^x \sin x - \frac{1}{2} e^x \cos x + C.$$

Problem 5. ($\star\star\star$) Compute

$$\int x e^x \cos(x) \, dx.$$

Solution 5.

Step 1: Draw the table

$\pm$	D	I
$+$	$x\cos x$	e^x
$-$	$\cos x - x\sin x$	e^x
$+\int$	$-2\sin x - x\cos x$	e^x

Step 2: From the table, we have

$$\int xe^x \cos x\,dx = xe^x\cos x - e^x\cos x + xe^x \sin x - 2\int e^x \sin x\,dx - \int xe^x\cos x\,dx.$$

Moving all the $\int xe^x \cos x\,dx$ to one side and simplifying, we can conclude

$$\begin{aligned}2\int xe^x\cos x\,dx &= xe^x\cos x - e^x\cos x + xe^x\sin x - 2\int e^x\sin x\,dx \\ &= xe^x\cos x - e^x\cos x + xe^x\sin x - e^x\sin x + e^x\cos x + C. \qquad \text{Problem 4}\end{aligned}$$

Dividing both sides by 2, we can conclude

$$\int xe^x\cos x\,dx = \frac{1}{2}\Big(xe^x\cos x + xe^x\sin x - e^x\sin x\Big) + C.$$

Problem 6. ($\star$) Compute

$$\int \ln(x)\,dx.$$

Solution 6.

Step 1: Draw the table

$\pm$	D	I
$+$	$\ln(x)$	1
$-\int$	$\frac{1}{x}$	x

Step 2: From the table, we have

$$\int \ln(x)\,dx = x\ln(x) - \int 1\,dx = x\ln(x) - x + C.$$

Problem 7. ($\star\star$) Compute

$$\int_1^2 x^3 \ln x\,dx.$$

Solution 7.

Step 1: Draw the table

$\pm$	D	I
$+$	$\ln x$	x^3
$-\int$	$\frac{1}{x}$	$\frac{1}{4}x^4$

Step 2: From the table, we have

$$\int x^3 \ln x\, dx = \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3\, dx = \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C.$$

Step 3: We can now use the fundamental theorem of calculus to compute the definite integral,

$$\int_1^2 x^3 \ln x\, dx = \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4\Big|_{x=1}^{x=2} = 4\ln 2 - 1 + \frac{1}{16} = 4\ln 2 - \frac{15}{16}.$$

Problem 8. ($\star\star\star$) Prove the reduction formula

$$\int \sin^n(x)\, dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\, dx.$$

Solution 8.

Step 1: Draw the table

$\pm$	D	I
$+$	$\sin^{n-1}(x)$	$\sin(x)$
$-\int$	$(n-1)\cos(x)\sin^{n-2}(x)$	$-\cos(x)$

Step 2: From the table, we have

$$\begin{aligned}\int \sin^n(x)\, dx &= -\sin^{n-1}(x)\cos(x) + (n-1)\int \cos^2(x)\sin^{n-2}(x)\\ &= -\sin^{n-1}(x)\cos(x) + (n-1)\int (1-\sin^2(x))\sin^{n-2}(x) && \sin^2(x)+\cos^2(x)=1\\ &= -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\, dx - (n-1)\int \sin^n(x)\, dx\end{aligned}$$

Moving all the the $\int \sin^n(x)\, dx$ terms to one side, we can conclude

$$\begin{aligned} n\int \sin^n(x)\, dx &= -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\, dx\\ \Rightarrow \int \sin^n(x)\, dx &= -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\, dx.\end{aligned}$$

1.6.1 Proofs of Properties of Integration

Problem 1. ($\star\star\star$) Suppose that $f(-x) = f(x)$. Prove that

$$\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx.$$

Solution 1. By the properties of definite integrals, we have

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx.$$

Using the change of variables $u = -x$ on the first integral, for even function f,

$$\begin{aligned}
\int_{0}^{-a} f(x)\,dx &= -\int_{0}^{a} f(-u)\,du && u=-x,\ du=-dx,\ x=0\to u=0,\ x=-a\to u=a \\
&= -\int_{0}^{a} f(u)\,du && f(-x)=f(x) \\
&= -\int_{0}^{a} f(x)\,dx.
\end{aligned}$$

This computation implies

$$\int_{-a}^{a} f(x)\,dx = -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

Problem 2. ($\star\star\star$) Suppose that $f(-x) = -f(x)$. Prove that

$$\int_{-a}^{a} f(x)\,dx = 0.$$

Solution 2. By the properties of definite integrals, we have

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx.$$

Using the change of variables $u = -x$ on the first integral, for odd functions f,

$$\begin{aligned}
\int_{0}^{-a} f(x)\,dx &= -\int_{0}^{a} f(-u)\,du && u=-x,\ du=-dx,\ x=0\to u=0,\ x=-a\to u=a \\
&= \int_{0}^{a} f(u)\,du && f(-x)=-f(x) \\
&= \int_{0}^{a} f(x)\,dx.
\end{aligned}$$

This computation implies

$$\int_{-a}^{a} f(x)\,dx = -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0.$$

Problem 3. ($\star\star\star$) Justify the technique used to solve separable ordinary differential equations:

$$\frac{dy}{dx} = f(x)g(y) \implies \int \frac{dy}{g(y)} = \int f(x)\,dx \Rightarrow G(y) = F(x) + C$$

where $G(y)$ is an antiderivative of $\frac{1}{g(y)}$ and $F(x)$ is an antiderivative of $f(x)$.

Solution 3. Using the notation $y'(x) = \frac{dy}{dx}$ and writing $y = y(x)$ explicitly as a function of x, we have

$$\frac{dy}{dx} = f(x)g(y) \Rightarrow \frac{y'(x)}{g(y(x))} = f(x) \Rightarrow \int \frac{y'(x)}{g(y(x))}\,dx = \int f(x)\,dx.$$

Using the change of variables $u = y(x)$ on the first integral involving the $y(x)$ term, we see

$$\int \frac{y'(x)}{g(y(x))}\,dx = \int \frac{du}{g(u)} = \int \frac{dy}{g(y)}.$$

Therefore, using this change of variables, we can conclude that

$$\frac{dy}{dx} = f(x)g(y) \implies \int \frac{dy}{g(y)} = \int f(x)\,dx.$$

This means there is a hidden change of variables that goes on when we formally separated $\frac{dy}{dx}$ in the second step of the technique.

Problem 4. $(\star\star\star)$ *Vanishing Theorem:* Suppose that $f(x)$ is a continuous function on $[a, b]$ such that $f(x) \geq 0$ for all $[a, b]$ and

$$\int_a^b f(x)\,dx = 0.$$

Prove that $f(x) = 0$ for all $x \in [a, b]$.

Solution 4. On the contrary, suppose that $f(x^*) > 0$ for some point $x^* \in [a, b]$. Then by continuity, we must also have that $f(x) > 0$ on some interval $[k, \ell] \subset [a, b]$. By the mean value theorem of integration, there exists a $c \in [k, \ell]$ such that

$$\int_k^\ell f(x)\,dx = f(c)(\ell - k).$$

Since we also have that $f(x) > 0$ for all $x \in [k, \ell]$, we must have $f(c) > 0$, which implies that

$$\int_k^\ell f(x)\,dx = f(c)(\ell - k) > 0.$$

Since $f(x) \geq 0$, by the monotonicity of integration, the conclusion above implies

$$\int_a^b f(x)\,dx \geq \int_k^\ell f(x)\,dx > 0,$$

which contradicts the fact that $\int_a^b f(x)\,dx = 0$. Therefore, we must have that $f(x) = 0$ for all $x \in [a, b]$.

Problem 5. $(\star\star\star)$ Prove the Mean Value Theorem for Integration by applying the Mean Value Theorem to the function $F(x) = \int_a^x f(t)\,dt$.

Solution 5. By the fundamental theorem of calculus, $F(x)$ is continuous on $[a, b]$ and $F(x)$ is differentiable on (a, b). Therefore, by the mean value theorem there exists a $c \in (a, b)$ such that

$$\frac{F(b) - F(a)}{b - a} = F'(c) = f(c).$$

Since $F(b) - F(a) = \int_a^b f(x)\,dx - \int_a^a f(x)\,dx = \int_a^b f(x)\,dx$, there exists a $c \in (a, b)$ such that

$$\frac{\int_a^b f(x)\,dx}{b - a} = f(c) \implies \int_a^b f(x)\,dx = f(c)(b - a).$$

Problem 6. ($\star\star\star$) *Jensen's Inequality:* Suppose that $f(x)$ is a continuous function on $[a, b]$ and $g''(x) \geq 0$ for all x. Prove that

$$g\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right) \leq \frac{1}{b-a}\int_a^b g(f(x))\,dx.$$

Solution 6. Let $L(x)$ be the linear approximation of g at the point $x = f_{ave}$. Since $g''(x) \geq 0$, we have $g(x)$ lies above its tangent line so,

$$L(x) := g(f_{ave}) + g'(f_{ave})(x - f_{ave}) \leq g(x).$$

Since this holds for all x, we have $L(f(x)) \leq g(f(x))$. Taking the average integral of both sides and using the monotonicity of definite integrals, we can conclude

$$g(f_{ave}) = \frac{1}{b-a}\int_a^b g(f_{ave}) + g'(f_{ave})(f(x) - f_{ave})\,dx \leq \frac{1}{b-a}\int_a^b g(f(x))\,dx.$$

Remark: We only require $g''(x) \geq 0$ on the range of f for the inequality to hold. This is because f_{ave} is in the range of f by the mean value theorem of integration, and g is only evaluated at points in the range of f. If $g''(x) > 0$, then the same proof shows $g(f_{ave}) < (g \circ f)_{ave}$ provided that f is not a constant function.

Remark: It is easy to see that the opposite inequality holds if $g''(x) \leq 0$. Suppose that $g''(x) \leq 0$, then $-g''(x) \geq 0$, so Jensen's inequality implies that

$$-g(f_{ave}) \leq (-g \circ f)_{ave} \implies g(f_{ave}) \geq (g \circ f)_{ave}.$$