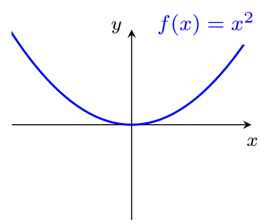
1 Functions and Inverses

Definition 1. A *function* $f : D \to R$ is a rule that assigns each element x in a set D to exactly one element $f(x)$ in R. The set D is called the *domain* of f. The set R is called the *range* of f, and it consists of all possible values of $f(x)$[1]. If the domain of a function is not explicitly specified, then we take D to be the largest set such that the function is well-defined.

Example 1. Functions can be represented in several ways such as a formula, a graph, or a table of values. For example, the relation $x \mapsto x^2$ from $\mathbb{R} \to [0, \infty)$ can be expressed as:

1. Formula: A mathematical formula $f(x) = x^2$
2. Graph: A *graph* is the collection of ordered pairs $\{(x, f(x)) : x \in D\}$ expressed as a curve in the xy-plane.



Note: A curve in the xy-plane is the graph of a function if any vertical line intersects the curve at most once. This is called the *vertical line test.*

3. Table: A table of values

x	-5	-4	-3	-2	-1	0	1	2	3	4
x^2	25	16	9	4	1	0	1	4	9	16

Definition 2. A function is *one-to-one* if it never takes the same value twice; that is, for every $x_1, x_2 \in D$,

$$f(x_1) = f(x_2) \implies x_1 = x_2. \tag{1}$$

Note: In the xy-plane, a function f is one-to-one if any horizontal line intersects the graph of f at most once. This is called the *horizontal line test.*

Definition 3. Let $f(x)$ and $g(x)$ be two one-to-one functions. The functions $f(x)$ and $g(x)$ are *inverses* if $(f \circ g)(x) = x$ for all x in the domain of g and $(g \circ f)(x) = x$ for all x in the domain of f. The function g is called the *inverse function of* f and is usually denoted by $g(x) = f^{-1}(x)$.

The inverse f^{-1} satisfies several properties:

1. The domain of f^{-1} is the range of f and the range of f^{-1} is the domain of f
2. In the xy-plane, the graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$
3. A point (x, y) is on the graph of f if and only if (y, x) is a point on the graph of f^{-1}.

Example 2. The functions $f(x) = x^2$ from $(0, \infty) \to (0, \infty)$ is a one-to-one function and has an inverse given by $f^{-1}(x) = g(x) = \sqrt{x}$. To verify this, we can check that $(f \circ g)(x) = (\sqrt{x})^2 = x$ for all $x \in (0, \infty)$ and $(g \circ f)(x) = \sqrt{x^2} = x$ for all $x \in (0, \infty)$.

[1] More generally, the set R in the notation $f : D \to R$ refers to the *co-domain* of f. The co-domain must contain the possible values of $f(x)$, but may also contain some impossible values. For example, one may write: "let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^2$" even though the range of f is only $[0, \infty)$. For simplicity, we always take the set R to refer to the range of a function in this course, so it is not important to know what a co-domain is.

1.1 Example Problems

1.1.1 Check if a function is one-to-one

Strategy: We need to check Definition 2 directly.

1. To show a function is one-to-one, we need to check condition (1) in Definition 2 holds by assuming $f(x_1) = f(x_2)$ and showing that we must have $x_1 = x_2$.
2. Alternatively, to show a function is one-to-one, it suffices to show that the function is strictly increasing or decreasing on its domain (i.e. $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ or $f(x_1) > f(x_2)$ whenever $x_1 < x_2$). This essentially shows the contrapositive of (1), i.e. if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$. This technique will be useful later on in this course once we build the tools to check this condition.
3. To show that a function is not one-to-one, it suffices to find points $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$.

Problem 1. ($\star$) Is the function $f(x) = \frac{x+5}{x-6}$ one to one?

Solution 1. To show the function is one-to-one, notice for $x_1 \neq 6$ and $x_2 \neq 6$, we have

$$\begin{aligned} f(x_1) = \frac{x_1+5}{x_1-6} = \frac{x_2+5}{x_2-6} = f(x_2) &\Rightarrow (x_1+5)(x_2-6) = (x_2+5)(x_1-6) \\ &\Rightarrow x_1x_2 - 6x_1 + 5x_2 - 30 = x_1x_2 - 6x_2 + 5x_1 - 30 \\ &\Rightarrow -11x_1 = -11x_2 \\ &\Rightarrow x_1 = x_2, \end{aligned}$$

so our function is one-to-one.

Problem 2. ($\star$) Is the function $f(x) = |x|$ one to one?

Solution 2. From Table 2, we know that $|x|$ is not one-to-one function. To show this directly, notice that

$$f(1) = |1| = 1 = |-1| = f(-1),$$

so our function is not one-to-one.

Problem 3. ($\star\star$) Is the function $f(x) = xe^{x^2}$ one to one?

Solution 3. We will show the function is one-to-one by proving that is strictly increasing. Taking the derivative of $f(x)$, we have

$$f'(x) = e^{x^2} + 2x^2e^{x^2} = e^{x^2}(1+x^2) > 0 \text{ for all } x \in \mathbb{R}.$$

In particular, we have $f(x)$ is strictly increasing on $\mathbb{R}$ and therefore also one-to-one.

Problem 4. ($\star\star$) True of False: If $f(x)$ is an even function (i.e. $f(-x) = f(x)$), then $f(x)$ is not one-to-one.

Solution 4. This is true. Since $f(-x) = f(x)$ for all $x \in D_f$, if we take $x \in D_f$ such that $x \neq 0$ then we have

$$f(x) = f(-x)$$

but $x \neq -x$ so our function is not one-to-one.

Problem 5. ($\star\star$) True of False: If $f(x)$ is an odd function (i.e. $f(-x) = -f(x)$), then $f(x)$ is one-to-one.

Solution 5. This is false. For example, consider $f(x) = \sin(x)$. It is well known that $\sin(x)$ is an odd function, but

$$f(0) = \sin(0) = 0 = \sin(2\pi) = f(2\pi),$$

so $f(x)$ is not one-to-one.

Problem 6. ($\star\star$) True or False: If $g : A \to B$ and $f : B \to C$ are one-to-one then $f \circ g : A \to C$ is one-to-one.

Solution 6. This is true. We can show Definition 2 directly. Suppose that $(f \circ g)(x_1) = (f \circ g)(x_2)$, we need to show that $x_1 = x_2$. Notice that both f and g satisfy (1) since they are one-to-one, so

$$\begin{aligned} (f \circ g)(x_1) = (f \circ g)(x_2) &\Rightarrow g(x_1) = g(x_2) && f \text{ satisfies } (1) \\ &\Rightarrow x_1 = x_2 && g \text{ satisfies } (1) \end{aligned}$$

as required.

1.1.2 Finding the Inverse of a Function

Strategy: Solve $x = f(y)$ for y. The resulting equation is $y = f^{-1}(x)$.

Problem 1. ($\star$) Find the formula for the inverse of the function $f(x) = \frac{x+5}{x-6}$.

Solution 1. It is easy to check that $f(x)$ is one-to-one. To find the formula for f^{-1}, it suffices to solve

$$x = f(y) = \frac{y+5}{y-6}$$

in terms of y. Notice that

$$x = \frac{y+5}{y-6} \Rightarrow xy - 6x = y + 5 \Rightarrow (x-1)y = 6x + 5 \Rightarrow f^{-1}(x) = y = \frac{6x+5}{x-1}.$$

Problem 2. ($\star$) Find the formula for the inverse of the function $f(x) = x^2 + 2x + 1$.

Solution 2. The function $f(x)$ is not one-to-one since $f(0) = 1 = f(-2)$, so the inverse does not exist.

Problem 3. ($\star\star$) Find a formula for the inverse of the function $f(x) = \sqrt{-1-x}$. Be sure to specify the domain of f^{-1}.

Solution 3. It is easy to check that $f(x)$ is one-to-one on its domain $(-\infty, -1]$. To find the formula for f^{-1}, it suffices to solve

$$x = f(y) = \sqrt{-1-y}$$

in terms of y. Notice that

$$x = \sqrt{-1-y} \Rightarrow x^2 = -1 - y \Rightarrow f^{-1}(x) = y = -1 - x^2.$$

Although the function $f^{-1}(x)$ makes sense for all $x \in \mathbb{R}$, the domain of this inverse is the range of $f(x)$, which is $[0, \infty)$.

2 Exponential Functions

An exponential function with base $a > 0$ is denoted by a^x and its inverse is denoted by $\log_a(x)$. The most common base for an exponential function is the mathematical constant $e = 2.718\ldots$

Definition 4. We call function $\exp(x) = e^x$ the *exponential function*, and its inverse $\ln(x) = \log_e(x)$ the *natural logarithm* (some books use $\log(x)$ to refer to the natural logarithm).

We summarize several properties of the exponential function and its logarithm. The following properties hold more generally for $a > 0$ instead of e (just replace the e with a). We will prove some change of base formulas in the exercises that express exponential functions and logarithms with base a in terms of $\exp(x)$ and $\ln(x)$. Since we can freely convert into base e, it suffices to just work with base e in most applications.

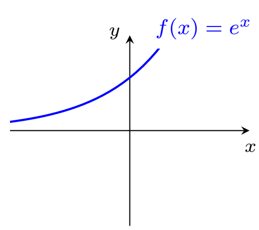
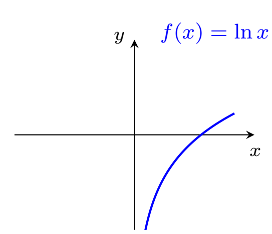
Exponential	**Logarithm**
$\exp(x+y) = \exp(x)\cdot\exp(y)$ $\exp(x)^y = \exp(y\cdot x)$ $\exp(-x) = \frac{1}{\exp(x)}$ $\exp(\ln(x)) = x$ for $x > 0$ $\exp(0) = 1, \quad \exp(1) = e$	$\ln(x\cdot y) = \ln(x) + \ln(y)$ $\ln(x^y) = y\ln(x)$ $-\ln(x) = \ln(\frac{1}{x})$ $\ln(\exp(x)) = x$ for $x \in \mathbb{R}$ $\ln(1) = 0, \quad \ln(e) = 1$
	

Table 1: Properties of Exponential Functions

Note: Some exponential expressions are defined for $a < 0$. For example, $(-8)^2 = 64$ and $(-8)^{1/3} = -2$. However, $(-8)^x$ is not defined on $\mathbb{R}$ (for example, $(-8)^{1/2}$ does not have a real solution).

2.1 Example Problems:

2.1.1 Solving and Simplifying Expressions Involving $\log(x)$ and e^x:

Problem 1. ($\star$) Solve $3^{3x} = \frac{1}{9^{2x-1}}$ for x.

Solution 1. Using the properties of exponents, we have

$$\begin{aligned} 3^{3x} = \frac{1}{9^{2x-1}} &\Rightarrow 3^{3x} = (3^2)^{-(2x-1)} \\ &\Rightarrow 3^{3x} = 3^{-2(2x-1)} \\ &\Rightarrow 3x = -2(2x-1) \\ &\Rightarrow x = \frac{2}{7}. \end{aligned}$$

Problem 2. ($\star$) Simplify the expression $e^{-\ln(\frac{1}{3})}$.

Solution 2. Using the properties of exponents, we have

$$e^{-\ln(\frac{1}{3})} = \exp\Big(-\ln\Big(\frac{1}{3}\Big)\Big) = \exp(\ln(3)) = 3.$$

Problem 3. ($\star$) Simplify the expression

$$e^{x\ln(x)+(2x-7)\ln(x)}.$$

Solution 3. Using the properties of exponents, we have

$$e^{x\ln(x)+(2x-7)\ln(x)} = e^{3x\ln(x)-7\ln(x)} = e^{\ln(x^{3x})+\ln(x^{-7})} = e^{\ln(x^{3x}\cdot x^{-7})} = e^{\ln(x^{(3x-7)})} = x^{3x-7}.$$

Problem 4. ($\star$) Solve the following expression for x:

$$\log_{54}(x-2) + \log_{54}(x+1) = 1.$$

Solution 4. Simplifying the left hand side, we combine the logarithms to conclude

$$\log_{54}(x-2) + \log_{54}(x+1) = \log_{54}((x-2)\cdot(x+1)) = \log_{54}(x^2 - x - 2).$$

Therefore, raising both sides of our original equation with base 54 implies

$$\log_{54}(x-2) + \log_{54}(x+1) = 1 \Rightarrow \log_{54}(x^2 - x - 2) = 1 \Rightarrow x^2 - x - 2 = 54 \Rightarrow x^2 - x - 56 = 0.$$

Notice that $x^2 - x - 56 = (x-8)(x+7)$ which means the equation $x^2 - x - 56 = 0$ has solutions $x = 8$ and $x = -7$. However, since the domain of $\log_{54}(x)$ is $x > 0$, we have $x = -7$ is not in the domain of either $\log_{54}(x-2)$ or $\log_{54}(x+1)$, so our only solution is $x = 8$.

Problem 5. ($\star\star$) Solve the following expression for x:

$$2^{x-1}\cdot 3^{x+1} = 54.$$

Solution 5. Taking the natural logarithm of both sides, we have

$$2^{x-1}\cdot 3^{x+1} = 54 \Rightarrow (x-1)\ln(2) + (x+1)\ln(3) = \ln(54)$$

Since $54 = 2\cdot 27 = 2\cdot 3^3$, we have

$$(x-1)\ln(2) + (x+1)\ln(3) = \ln(54) = \ln(2\cdot 3^3) = \ln(2) + 3\ln(3).$$

Simplifying this equation, we have

$$x\ln(2) + x\ln(3) = 2\ln(2) + 2\ln(3) \Rightarrow x = \frac{2\ln(2) + 2\ln(3)}{\ln(2) + \ln(3)} = 2.$$

2.1.2 Change of Base Formulas:

Problem 1. ($\star\star$) Let $b > 0$ and suppose $b \neq 1$. Prove the change of base formula

$$\log_b(x) = \frac{\ln x}{\ln b}.$$

Solution 1. We solve the following equation for y

$$\begin{aligned} y = \log_b(x) &\Leftrightarrow b^y = x && \text{compose both sides with } b^x \\ &\Leftrightarrow \ln(b^y) = \ln(x) && \text{compose both sides with } \ln(x) \\ &\Leftrightarrow y\ln(b) = \ln(x) \\ &\Leftrightarrow y = \frac{\ln(x)}{\ln(b)}. \end{aligned}$$

Therefore, we have

$$\log_b(x) = y = \frac{\ln(x)}{\ln(b)}.$$

Note: We used the fact $b \neq 1$, to ensure we did not divide by 0.

Problem 2. ($\star\star$) Prove the change of base formula

$$b^x = e^{x\ln(b)}.$$

Solution 2. We solve the following equation for y

$$\begin{aligned} b^x = e^y &\Leftrightarrow \ln b^x = y && \text{compose both sides with } \ln(x) \\ &\Leftrightarrow y = x\ln(b). \end{aligned}$$

Therefore, we have

$$b^x = e^y = e^{x\ln(b)}.$$

3 Absolute Value

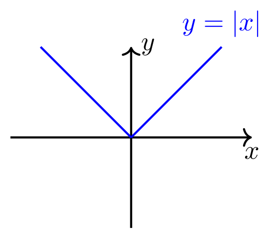
For $x \in \mathbb{R}$, the absolute value of x is a piecewise function defined by

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}.$$

It is easy to check that for $x, y \in \mathbb{R}$, we have

1. Non-negativity: $|x| \geq 0$
2. Multiplicativity: $|xy| = |x||y|$
3. Positive Definiteness: $|x| = 0$ if and only if $x = 0$
4. Triangle Inequality: $|x + y| \leq |x| + |y|$
5. Reverse Triangle Inequality: $||x| - |y|| \leq |x - y|$.

The graph is given below:



3.1 Example Problems

3.1.1 Absolute Value Inequalities:

Strategy: We can proceed in two ways:

1. In general, we need to break our region into cases where our absolute values change sign and solve the inequality on each region separately.
2. Shortcut: If we want to compute $|f(x)| \leq C$ or $|f(x)| \geq C$ then we can replace the absolute value with $\pm$ and solve the two cases corresponding to $+f(x)$ and $-f(x)$.

Problem 1. ($\star$) Find all x such that

$$|2x - 4| \leq |x + 3|.$$

Solution 1. Rearranging the inequality, we have

$$|2x - 4| \leq |x + 3| \Rightarrow \left|\frac{2x - 4}{x + 3}\right| \leq 1 \Rightarrow \pm\frac{2x - 4}{x + 3} \leq 1.$$

Solving for the case with the positive sign, we have

$$\frac{2x - 4}{x + 3} \leq 1 \Rightarrow 2x - 4 \leq x + 3 \Rightarrow x \leq 7$$

and for the case with the negative sign, we have

$$-\frac{2x-4}{x+3} \leq 1 \Rightarrow -2x+4 \leq x+3 \Rightarrow -3x \leq -1 \Rightarrow x \geq \frac{1}{3}.$$

Therefore, we have

$$\frac{1}{3} \leq x \leq 7.$$

Note: Notice that $x \neq -3$, so we do not run into the issue of dividing by zero in the first step.

Problem 2. ($\star$) Find the domain of the function

$$f(x) = \sin^{-1}(x^3 - 7).$$

Solution 2. Since the domain of $\sin^{-1} x$ is $|x| \leq 1$, the domain of $f(x)$ are the values of x such that $|x^3 - 7| \leq 1$. To solve this inequality, we notice

$$|x^3 - 7| \leq 1 \Rightarrow \pm(x^3 - 7) \leq 1$$

Solving for the case with the positive sign, we have

$$(x^3 - 7) \leq 1 \Rightarrow x^3 \leq 8 \Rightarrow x \leq 2$$

and for the case with the negative sign, we have

$$-(x^3 - 7) \leq 1 \Rightarrow -x^3 \leq -6 \Rightarrow x \geq 6^{1/3}.$$

Therefore, we have

$$6^{1/3} \leq x \leq 2.$$

Problem 3. ($\star$) Find all x such that

$$|x + 8| < 5x + 10.$$

Solution 3. The function $|x + 8|$ changes sign when $x = -8$, so we consider the regions $x < -8$ and $x > -8$.

1. $x > -8$: In this case we have $|x + 8| = x + 8$, so solving the inequality gives

   $$|x + 8| < 5x + 10 \Rightarrow x + 8 < 5x + 10 \Rightarrow x > -\frac{1}{2}.$$

   Since we must have both $x \geq -8$ and $x > -\frac{1}{2}$, we have our inequality is satisfied when $x > -\frac{1}{2}$.

2. $x < -8$: In this case we have $|x + 8| = -(x + 8)$ so solving the inequality gives

   $$|x + 8| < 5x + 10 \Rightarrow -x - 8 < 5x + 10 \Rightarrow x > -3.$$

   Since we must have both $x < -8$ and $x > -3$, no x satisfies our inequality.

3. $x = 8$: When $x = -8$, we have $0 < -40 + 10$, so $x = -8$ does not satisfy our inequality.

Combining the cases above, our solutions are $x > -\frac{1}{2}$.

Problem 4. ($\star\star$) Find all x such that

$$|x-2| < |x+4| - 2.$$

Solution 4. The function $|x-2|$ changes sign when $x = 2$ and $|x+4|$ changes sign when $x = -4$, so we consider the cases

1. $x < -4$: On this region, we have $|x-2| = -x+2$ and $|x+4| = -x-4$ so we have

   $$|x-2| < |x+4| - 2 \Rightarrow -x+2 < -x-4-2 \Rightarrow 8 < 0,$$

   which is a false expression, so no x in this region satisfies our inequality.

2. $-4 < x < 2$: On this region, we have $|x-2| = -x+2$ and $|x+4| = x+4$ so we have

   $$|x-2| < |x+4| - 2 \Rightarrow -x+2 < x+4-2 \Rightarrow x > 0.$$

   so we must have $x > 0$ and $-4 < x < 2$ which means $0 < x < 2$ is a solution to the inequality.

3. $x > 2$: On this region, we have $|x-2| = x-2$ and $|x+4| = x+4$ so we have

   $$|x-2| < |x+4| - 2 \Rightarrow x-2 < x+4-2 \Rightarrow 0 < 4,$$

   which is a true expression so $x > 2$ is a solution to the inequality.

4. $x = -4$ or $x = 2$: When $x = -4$, we have $6 < -2$ which is false, so $x = -4$ is not a solution. When $x = 2$, we have $0 < 4$ which is true, so $x = 2$ is a solution.

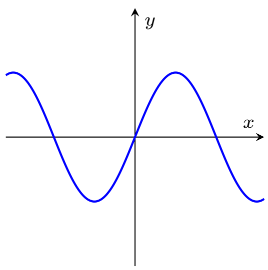
Combining our cases above, our solutions are $x > 0$.

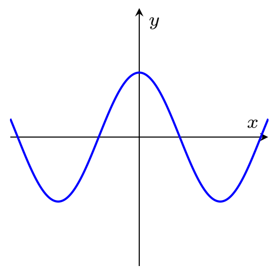
4 Trigonometric Functions

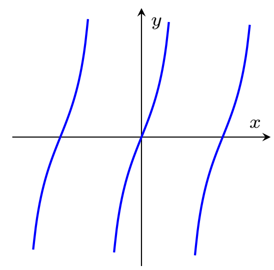
The 3 main trigonometric functions discussed in this course are $\sin(x)$, $\cos(x)$ and $\tan(x)$. The key values of $\sin(x)$ and $\cos(x)$ can be summarized by the table of values

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

The values for other key values can be extrapolated by remembering the shapes of the graphs







The key trigonometric identities are

1. Pythagorean Identity:
$$\sin^2\theta + \cos^2\theta = 1.$$

2. Sum and Difference Formulas:
$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \sin(\beta)\cos(\alpha), \qquad \cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta).$$

from these, one can derive the following identities

3. Symmetry and Periodicity: (Use the Sum and Difference Formulas)
$$\sin(-\theta) = -\sin(\theta), \quad \cos(-\theta) = \cos(\theta), \quad \sin(\theta + 2k\pi) = \sin(\theta), \quad \cos(\theta + 2k\pi) = \cos(\theta)$$

4. Complementary and Supplementary Angles: (Use the Sum and Difference Formulas)
$$\sin\Big(\frac{\pi}{2} - \theta\Big) = \cos(\theta), \quad \sin(\pi - \theta) = \sin(\theta), \quad \cos\Big(\frac{\pi}{2} - \theta\Big) = \sin(\theta), \quad \cos(\pi - \theta) = -\cos(\theta).$$

5. Double Angle Formulas: (Use the Sum and Difference Formulas and the Pythagorean Identity)
$$\sin(2\theta) = 2\sin(\theta)\cos(\theta), \qquad \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 1 - 2\sin^2(\theta) = 2\cos^2(\theta) - 1.$$

6. Half Angle Formulas: (Use the Double Angle Formulas for $\cos(2\theta)$)
$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}, \qquad \cos^2\theta = \frac{1 + \cos 2\theta}{2}.$$

7. Product to Sum Formulas: (Use the Sum and Difference Formulas)
$$\cos(\alpha)\cos(\beta) = \frac{1}{2}(\cos(\alpha + \beta) + \cos(\alpha - \beta)), \quad \sin(\alpha)\sin(\beta) = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta)),$$
$$\sin(\alpha)\cos(\beta) = \frac{1}{2}(\sin(\alpha + \beta) + \sin(\alpha - \beta)).$$

To solve some word problems, it is also useful to recall the *Cosine Law*
$$c^2 = a^2 + b^2 - 2ab\cos C \quad \text{where } C \text{ is the angle opposite side } c.$$

4.1 Example Problems

4.1.1 General Trigonometry Problems

Problem 1. ($\star$) Find all x such that

$$\cos\Big(x + \frac{\pi}{2}\Big) = 0.$$

Solution 1. From the graph of $\cos(x)$, we know $\cos(x) = 0 \Rightarrow x = \frac{\pi}{2} + k\pi$ for $k \in \mathbb{Z}$. Therefore, the solutions of our equation are x such that

$$x + \frac{\pi}{2} = \frac{\pi}{2} + k\pi \Rightarrow x = k\pi \text{ for } k \in \mathbb{Z}.$$

Problem 2. ($\star\star\star$) Derive the Sum and Difference formulas.

Solution 2. Recall Euler's Identity,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

If we take $\theta = \alpha + \beta$, then

$$e^{i(\alpha+\beta)} = \cos(\alpha + \beta) + i\sin(\alpha + \beta)$$

and using the fact $\exp(a + b) = \exp(a)\exp(b)$, we also have

$$\begin{aligned} e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta} &= (\cos(\alpha) + i\sin(\alpha))(\cos(\beta) + i\sin(\beta)) \\ &= \Big(\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)\Big) + i\Big(\sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)\Big). \end{aligned}$$

Therefore, our two equations above implies

$$\cos(\alpha+\beta)+i\sin(\alpha+\beta) = e^{i(\alpha+\beta)} = \Big(\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)\Big)+i\Big(\sin(\alpha)\cos(\beta)+\sin(\beta)\cos(\alpha)\Big).$$

Equating the real and imaginary parts, we have

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \text{ and } \sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha).$$

To derive the formulas for $\alpha - \beta$, we use the fact $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$ and use our formulas above to conclude

$$\cos(\alpha - \beta) = \cos(\alpha + (-\beta)) = \cos(\alpha)\cos(-\beta) - \sin(\alpha)\sin(-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

and

$$\sin(\alpha - \beta) = \sin(\alpha + (-\beta)) = \sin(\alpha)\cos(-\beta) + \sin(-\beta)\cos(\alpha) = \sin(\alpha)\cos(\beta) - \sin(\beta)\cos(\alpha).$$

4.1.2 Inverse Trig Problems

Problem 1. ($\star$) Find the x such that

1. $\sin(\sin^{-1}(x)) = x$
2. $\sin^{-1}(\sin(x)) = x$
3. $\tan^{-1}(\tan(x)) = x$
4. $\cot^{-1}(\cot(x)) = x$
5. $\csc(\csc^{-1}(x)) = x$
6. $\cos(\cos^{-1}(x)) = x$.

Solution 1. Recall that the domain of a composition $f \circ g$ is $\{x \in D_g : g(x) \in D_f\}$ where D_g and D_f are the respective domains of g and f. Even though the trigonometric functions $\sin(x), \cos(x), \dots$ are defined on large domains, the functions are not one-to-one, so its inverses are not well defined on the larger domain. We always restrict the domain of the trigonometric functions to the range of the inverse functions so that the inverses behave nicely.

The answers to these problems can be read directly off the domains and ranges of the inverse trigonometric functions in Table 2. Let D_f and R_f be the domains and ranges of f.

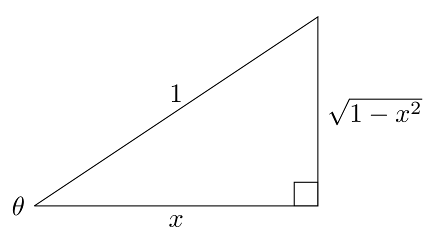
1. The inverse identity $\sin(\sin^{-1}(x)) = x$ holds for all $x \in D_{\sin^{-1}(x)} = [-1, 1]$.
2. The inverse identity $\sin^{-1}(\sin(x)) = x$ holds for all $x \in D_{\sin(x)} = R_{\sin^{-1}(x)} = [-\frac{\pi}{2}, \frac{\pi}{2}]$.
3. The inverse identity $\tan^{-1}(\tan(x)) = x$ holds for all $x \in D_{\tan(x)} = R_{\tan^{-1}(x)} = (-\frac{\pi}{2}, \frac{\pi}{2})$.
4. The inverse identity $\cot^{-1}(\cot(x)) = x$ holds for all $x \in D_{\cot(x)} = (0, \pi)$.
5. The inverse identity $\csc(\csc^{-1}(x)) = x$ holds for all $x \in D_{\csc^{-1}(x)} = (-\infty, -1] \cup [1, \infty)$.
6. The inverse identity $\cos(\cos^{-1}(x)) = x$ holds for all $x \in D_{\cos^{-1}(x)} = [-1, 1]$.

Problem 2. ($\star\star$) Rewrite the expression $\tan(\cos^{-1}(x))$ without using trigonometric functions. What is the domain of this function?

Solution 2. We can solve this problem either geometrically or algebraically.

Geometric Solution: We first find the domain of our function. We have $D_{\cos^{-1}(x)} = [-1, 1]$ and $D_{\tan(x)} = \{x \neq \frac{2k+1}{2}\pi\}$, so our domain consists of points in $D_{\cos^{-1}(x)}$ such that $\cos^{-1}(x) \neq \frac{\pi}{2} \Rightarrow x \neq 0$. Therefore, the domain of our function is $[-1, 1] \setminus \{0\}$.

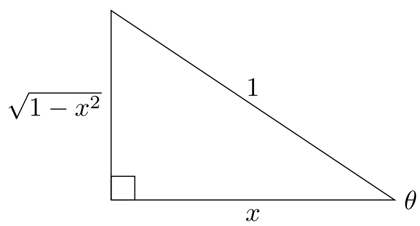
Case $x > 0$: We first consider the case such that $x > 0$ on our domain. On this region, we have $\theta = \cos^{-1}(x) \in [0, \frac{\pi}{2}]$ (the first quadrant). The triangle corresponding to $\cos(\theta) = x$ in the first quadrant is given by



From this triangle, we see

$$\tan(\cos^{-1}(x)) = \tan(\theta) = \frac{\sqrt{1-x^2}}{x} \text{ for } x \in (0, 1].$$

Case $x < 0$: We now consider the case such that $x < 0$ on our domain. On this region, we have $\theta = \cos^{-1}(x) \in [\frac{\pi}{2}, \pi]$ (the second quadrant). The triangle corresponding to $\cos(\theta) = x$ in the second quadrant is given by



Notice that $x < 0$, so this triangle is indeed in the second quadrant. From this triangle, we see

$$\tan(\cos^{-1}(x)) = \tan(\theta) = \frac{\sqrt{1-x^2}}{x} \text{ for } x \in [-1, 0).$$

Algebraic Solution: We first find the domain of our function. We have $D_{\cos^{-1}(x)} = [-1, 1]$ and $D_{\tan(x)} = \{x \neq \frac{2k+1}{2}\pi\}$, so our domain consists of points in $D_{\cos^{-1}(x)}$ such that $\cos^{-1}(x) \neq \frac{\pi}{2} \Rightarrow x \neq 0$. Therefore, the domain of our function is $[-1, 1] \setminus \{0\}$.

Case $x > 0$: We first consider the case such that $x \geq 0$ on our domain. On this region, we have $\theta = \cos^{-1}(x) \in [0, \frac{\pi}{2}]$ so trigonometric functions are positive. We now solve the identity algebraically.

We want to write $\tan(\theta)$ in terms of $\cos(\theta)$. Using the Pythagorean identity,

$$\begin{aligned} \sin^2(\theta) + \cos^2(\theta) = 1 \Rightarrow \tan^2(\theta) + 1 &= \frac{1}{\cos^2(\theta)} \\ \Rightarrow \tan^2(\theta) &= \frac{1 - \cos^2(\theta)}{\cos^2(\theta)} \\ \Rightarrow \tan(\theta) &= \frac{\sqrt{1-\cos^2(\theta)}}{\cos(\theta)}. \end{aligned}$$

Since $\tan(\theta) \geq 0$ and $\cos(\theta) > 0$, we didn't have to worry about absolute values when taking the squareroots of both sides or dividing by zero. Therefore, if we set $\theta = \cos^{-1}(x)$, we have

$$\tan(\cos^{-1}(x)) = \frac{\sqrt{1 - \cos^2(\cos^{-1}(x))}}{\cos(\cos^{-1}(x))} = \frac{\sqrt{1-x^2}}{x} \text{ for } x \in (0, 1].$$

Case $x < 0$: We now consider the case such that $x < 0$ on our domain. We can easily check that our function $\tan(\cos^{-1}(x))$ is odd. To see this, notice that $\cos^{-1}(x) - \frac{\pi}{2}$ is odd, and therefore $\tan((\cos^{-1}(x) - \frac{\pi}{2}) + \frac{\pi}{2})$ is a composition of odd functions and therefore odd. Extending our solution for $x > 0$ to make it odd, we have

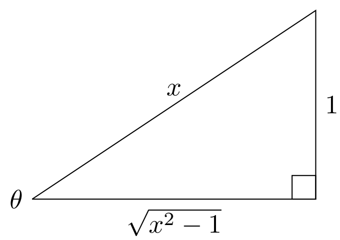
$$\tan(\cos^{-1}(x)) = -\frac{\sqrt{1-(-x)^2}}{-x} = \frac{\sqrt{1-x^2}}{x} \text{ for } x \in [-1, 0).$$

Problem 6. ($\star\star$) Rewrite the expression $\tan(\csc^{-1}(x))$ without using trigonometric functions. What is the domain of this function?

Solution 6. We can solve this problem either geometrically or algebraically.

Geometric Solution: We first find the domain of our function. We have $D_{\csc^{-1}(x)} = (-\infty, -1] \cup [1, \infty)$ and $D_{\tan(x)} = \{x \neq \frac{2k+1}{2}\pi\}$, so our domain consists of points in $D_{\csc^{-1}(x)}$ such that $\csc^{-1}(x) \neq \pm\frac{\pi}{2} \Rightarrow x \neq \pm 1$. Therefore, the domain of our function is $(-\infty, -1) \cup (1, \infty)$.

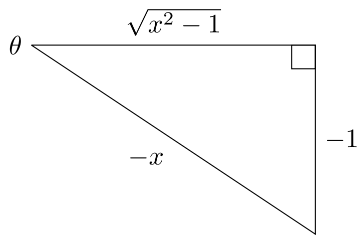
Case $x > 0$: We first consider the case such that $x > 0$ on our domain. On this region, we have $\theta = \csc^{-1}(x) \in [0, \frac{\pi}{2}]$ (the first quadrant). The triangle corresponding to $\csc(\theta) = x$ in the first quadrant is given by



From this triangle, we see

$$\tan(\csc^{-1}(x)) = \tan(\theta) = \frac{1}{\sqrt{x^2-1}} \text{ for } x \in (1,\infty).$$

Case $x<0$: We first consider the case such that $x>0$ on our domain. On this region, we have $\theta = \csc^{-1}(x) \in [-\frac{\pi}{2}, 0]$ (the fourth quadrant). The triangle corresponding to $\csc(\theta) = x$ in the fourth quadrant is given by



Notice that $x<0$, so the hypotenuse is positive. From this triangle, we see

$$\tan(\csc^{-1}(x)) = \tan(\theta) = -\frac{1}{\sqrt{x^2-1}} \text{ for } x \in (-\infty,-1).$$

Algebraic Solution: We first find the domain of our function. We have $D_{\csc^{-1}(x)} = (-\infty,-1] \cup [1,\infty)$ and $D_{\tan(x)} = \{x \neq \frac{2k+1}{2}\pi\}$, so our domain consists of points in $D_{\csc^{-1}(x)}$ such that $\csc^{-1}(x) \neq \pm\frac{\pi}{2} \Rightarrow x \neq \pm 1$. Therefore, the domain of our function is $(-\infty,-1) \cup (1,\infty)$.

Case $x>0$: We first consider the case such that $x>0$ on our domain. On this region, we have $\theta = \csc^{-1}(x) \in [0, \frac{\pi}{2}]$ so trig functions are all positive. We now solve the identity algebraically.

We want to write $\tan(\theta)$ in terms of $\csc(\theta)$. Using the Pythagorean identity,

$$\begin{aligned} \sin^2(\theta) + \cos^2(\theta) = 1 \Rightarrow 1 + \frac{1}{\tan^2(\theta)} &= \csc(\theta) \\ \Rightarrow \tan^2(\theta) &= \frac{1}{\csc^2(\theta)-1} \\ \Rightarrow \tan(\theta) &= \frac{1}{\sqrt{\csc^2(\theta)-1}}. \end{aligned}$$

Since $\sin(\theta) > 0$ and $\cos(\theta) > 0$ on this domain, we didn't have to worry about absolute values when taking the squareroots of both sides or dividing by zero. Therefore, if we set $\theta = \csc^{-1}(x)$, we have

$$\tan(\csc^{-1}(x)) = \frac{1}{\sqrt{x^2-1}} \text{ for } x \in (1,\infty).$$

Case $x<0$: We now consider the case such that $x<0$ on our domain. We can easily check that our function $\tan(\csc^{-1}(x))$ is odd. To see this, notice that $\csc^{-1}(x)$ is odd, and therefore $\tan((\csc^{-1}(x))$ is a composition of odd functions and therefore odd. Extending our solution to make it odd, we have

$$\tan(\csc^{-1}(x)) = -\frac{1}{\sqrt{(-x)^2-1}} = -\frac{1}{\sqrt{x^2-1}} \text{ for } x \in (-\infty,-1).$$

Remark: We used the following fact in Solutions 5 and Solution 6.

Suppose we know $f(x)$ for $x > 0$. We can use following formulas to extend our functions in an odd or even manner

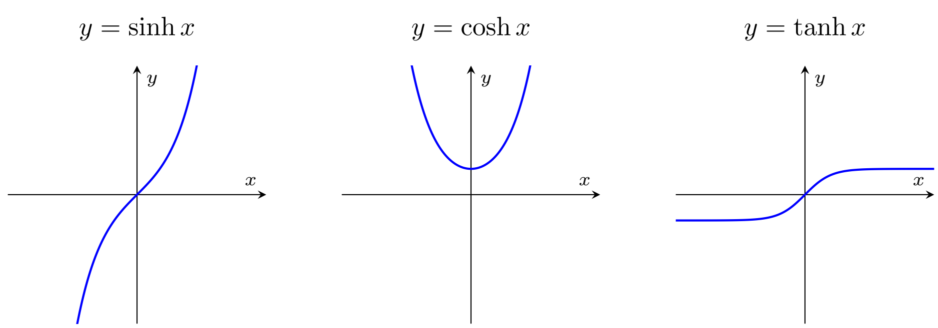
1. Odd Extension: For $x < 0$, the odd extension of f is given by $-f(-x)$.
2. Even Extension: For $x < 0$, the even extension of f is given by $f(-x)$.

5 Hyperbolic Functions

The 3 main hyperbolic functions discussed in this course are

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2}, \quad \tanh(x) = \frac{\sinh(x)}{\cosh(x)}.$$

The graphs of these functions are given by



Just like the trigonometric functions, the hyperbolic functions satisfy a similar set of properties

1. Analogue of the "Pythagorean" Identity:
$$\cosh^2(x) - \sinh^2(x) = 1.$$

2. Sum and Difference Formulas:
$$\sinh(x \pm y) = \sinh(x)\cosh(y) \pm \sinh(x)\cosh(y), \qquad \cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y).$$

3. Double Angle Formulas: (Use sum and difference formulas and the Pythagorean identity)
$$\sinh(2x) = 2\sinh(x)\cosh(x), \qquad \cosh(2x) = \cosh^2(x) + \sinh^2(x) = 2\sinh^2(x) + 1 = 2\cosh^2(x) - 1.$$

4. Half Angle Formulas: (Use the double angle formula for $\cosh(2x)$)
$$\sinh^2(x) = \frac{\cosh(2x) - 1}{2}, \qquad \cosh^2(x) = \frac{\cosh(2x) + 1}{2}.$$

Similarly to the trigonometric functions, we can also define the following lesser used hyperbolic functions
$$\operatorname{csch}(x) = \frac{1}{\sinh(x)}, \quad \operatorname{sech}(x) = \frac{1}{\cosh(x)}, \quad \coth(x) = \frac{1}{\tanh(x)}.$$

5.1 Example Problems

Problem 1. ($\star\star$) Derive the formula for $\cosh^{-1}(x)$ restricted to $x \geq 0$.

Solution W. e explicitly compute the inverse of $f(x) = \cosh(x)$. Since $\cosh(x)$ is not one to one, we have to restrict its domain to $x \geq 0$ to make our function one to one. We know that the range of

$\cosh(x)$ is $[1,\infty)$. Setting $f(y)=x$ and solving for y, we have

$$\begin{aligned}
\cosh(y) = x \Rightarrow \frac{e^y+e^{-y}}{2} &= x \\
&\Rightarrow e^y + e^{-y} - 2x = 0 \\
&\Rightarrow e^{2y} - 2xe^y + 1 = 0 && \text{multiply both sides by } e^y \\
&\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2-4}}{2} && \text{using the quadratic formula} \\
&\Rightarrow y = \ln(x \pm \sqrt{x^2-1}) \\
&\Rightarrow y = \ln(x + \sqrt{x^2-1}) && \text{since } y \text{ must be} > 0 \text{ for all } x \geq 1.
\end{aligned}$$

The other possible solution does not work because $\ln(x-\sqrt{x^2-1}) < 0$ for $x \geq 1$. Therefore, the formula for the inverse function is $f^{-1}(x) = \ln(x+\sqrt{x^2-1})$.

Note: If we multiplied both sides by e^{-y}, in the computation above, we would have deduced that $f^{-1}(x) = -\ln(x+\sqrt{x^2-1})$. This is the other possible inverse of $\cosh^{-1}(x)$ if we choose to restrict its domain to $x \leq 0$.

Problem 2. ($\star\star$) Verify the Pythagorean identity

$$\cosh^2(x) - \sinh^2(x) = 1.$$

Solution 2. This is a direct computation. We have

$$\cosh^2(x) - \sinh^2(x) = \Big(\frac{e^y+e^{-y}}{2}\Big)^2 + \Big(\frac{e^y-e^{-y}}{2}\Big)^2 = \frac{e^{2y}+2+e^{-2y}-e^{2y}+2-e^{-2y}}{4} = 1.$$

6 Appendix: Essential Functions and Their Graphs

Below is a non-exhaustive list of the basic functions we will encounter in this class.

Elementary Functions			
Function	Domain	Range	One-to-One
x^n (where n is even)	$\mathbb{R}$	$[0,\infty)$	No
x^n (where n is odd)	$\mathbb{R}$	$\mathbb{R}$	Yes
$\sqrt{x}$	$[0,\infty)$	$[0,\infty)$	Yes
$\frac{1}{x}$	$\mathbb{R}\setminus\{0\}$	$\mathbb{R}\setminus\{0\}$	Yes
$\lvert x\rvert$	$\mathbb{R}$	$[0,\infty)$	No
Exponential Functions			
Function	Domain	Range	One-to-One
a^x (where $a>0$)	$\mathbb{R}$	$(0,\infty)$	Yes
$\log_a(x)$ (where $a>0$)	$(0,\infty)$	$\mathbb{R}$	Yes
Trigonometric Functions			
Function	Domain	Range	One-to-One
$\sin(x)$	$\mathbb{R}$	$[-1,1]$	No
$\cos(x)$	$\mathbb{R}$	$[-1,1]$	No
$\tan(x)=\frac{\sin(x)}{\cos(x)}$	$\{x : x\neq \frac{2k+1}{2}\pi, k\in\mathbb{Z}\}$	$\mathbb{R}$	No
$\sin^{-1}(x)$	$[-1,1]$	$[-\frac{\pi}{2},\frac{\pi}{2}]$	Yes
$\cos^{-1}(x)$	$[-1,1]$	$[0,\pi]$	Yes
$\tan^{-1}(x)$	$\mathbb{R}$	$(-\frac{\pi}{2},\frac{\pi}{2})$	Yes
$\csc(x)=\frac{1}{\sin(x)}$	$\{x : x\neq k\pi, k\in\mathbb{Z}\}$	$(-\infty,-1]\cup[1,\infty)$	No
$\sec(x)=\frac{1}{\cos(x)}$	$\{x : x\neq \frac{2k+1}{2}\pi, k\in\mathbb{Z}\}$	$(-\infty,-1]\cup[1,\infty)$	No
$\cot(x)=\frac{1}{\tan(x)}$	$\{x : x\neq k\pi, k\in\mathbb{Z}\}$	$\mathbb{R}$	No
$\csc^{-1}(x)$	$(-\infty,-1]\cup[1,\infty)$	$[-\frac{\pi}{2},0)\cup(0,\frac{\pi}{2}]$	Yes
$\sec^{-1}(x)$	$(-\infty,-1]\cup[1,\infty)$	$[0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi]$	Yes
$\cot^{-1}(x)$	$\mathbb{R}$	$(0,\pi)$	Yes
Hyperbolic Functions			
Function	Domain	Range	One-to-One
$\sinh(x)=\frac{e^x-e^{-x}}{2}$	$\mathbb{R}$	$\mathbb{R}$	Yes
$\cosh(x)=\frac{e^x+e^{-x}}{2}$	$\mathbb{R}$	$[1,\infty)$	No
$\tanh(x)=\frac{\sinh(x)}{\cosh(x)}$	$\mathbb{R}$	$(-1,1)$	Yes
$\sinh^{-1}(x)=\ln(x+\sqrt{x^2+1})$	$\mathbb{R}$	$\mathbb{R}$	Yes
$\cosh^{-1}(x)=\ln(x+\sqrt{x^2-1})$	$[1,\infty)$	$[0,\infty)$	Yes
$\tanh^{-1}(x)=\frac{1}{2}\ln(\frac{1+x}{1-x})$	$(-1,1)$	$\mathbb{R}$	Yes
$\operatorname{csch}(x)=\frac{1}{\sinh(x)}$	$\mathbb{R}\setminus\{0\}$	$\mathbb{R}\setminus\{0\}$	Yes
$\operatorname{sech}(x)=\frac{1}{\cosh(x)}$	$\mathbb{R}$	$(0,1]$	No
$\coth(x)=\frac{1}{\tanh(x)}$	$\mathbb{R}\setminus\{0\}$	$(-\infty,-1)\cup(1,\infty)$	Yes

Table 2: Table of Functions

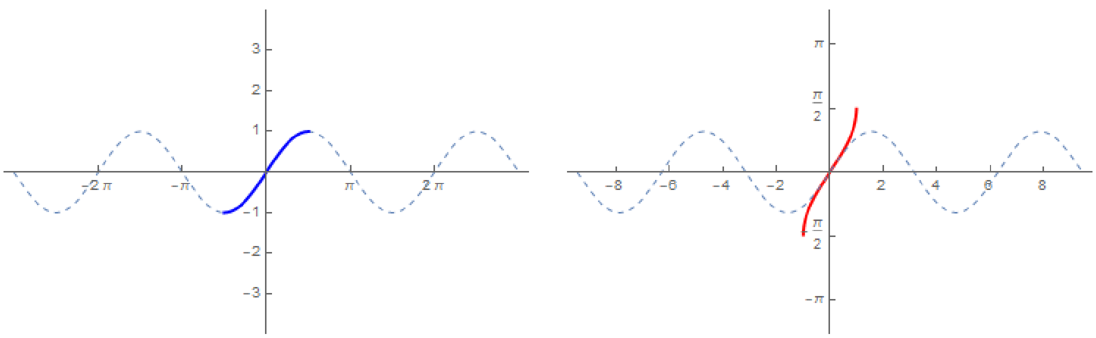
6.1 Graphs

6.1.1 Trigonometric Functions

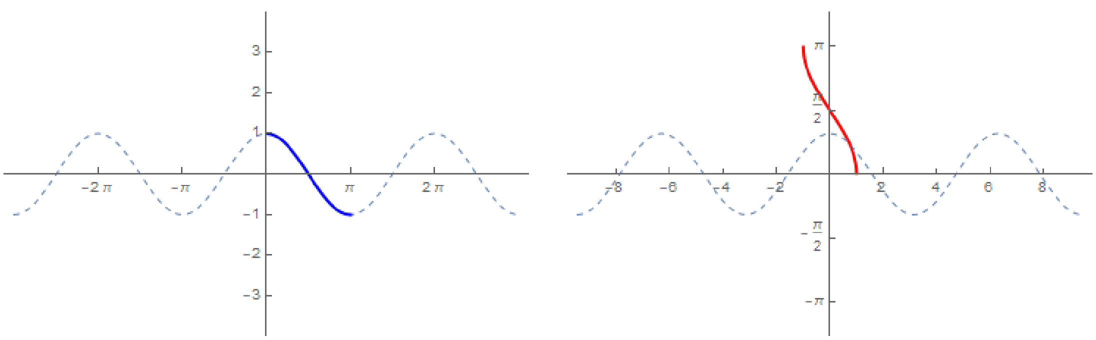
The trigonometric functions are not one-to-one, so we first restrict the domain of the trig functions to a region such that the functions is one-to-one. Trigonometric functions with "leading sine terms" are restricted to a subset of the domain $[-\pi/2, \pi/2]$ and the Trigonometric functions with "leading cosine terms" are restricted to the domain $[0, \pi]$.

Note: In the following pictures, the dotted blue graph is the full function. The blue graph is the one-to-one function on the restricted domain. And the red graph is the inverse function.

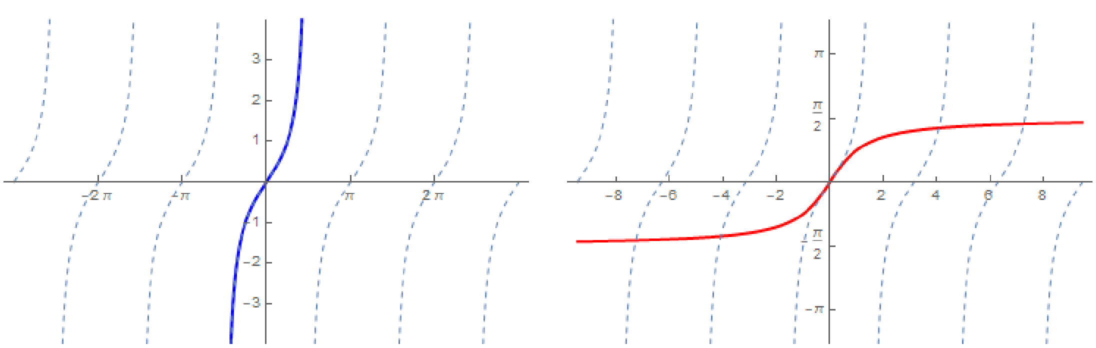
Sine Function:



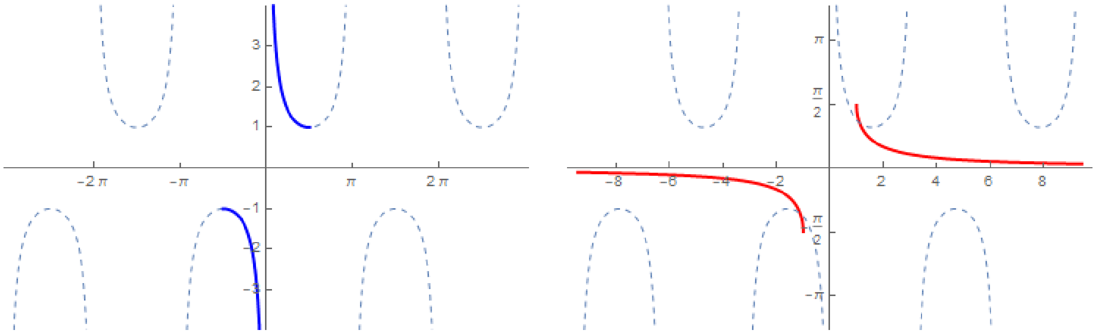
Cosine Function:



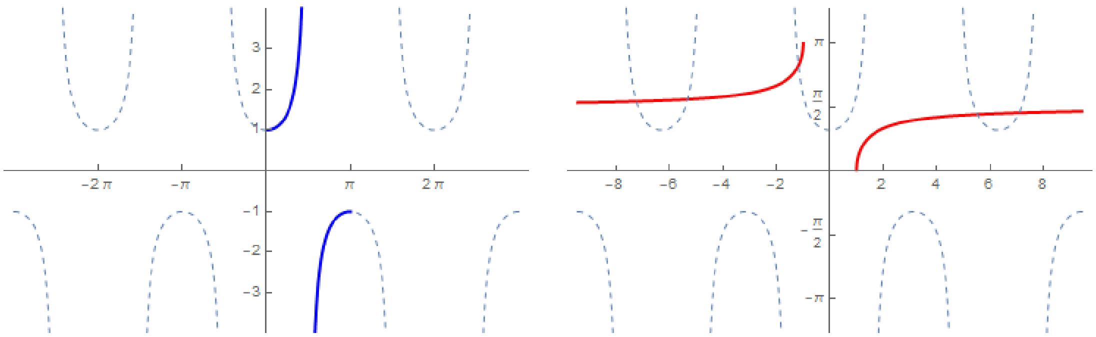
Tangent Function:



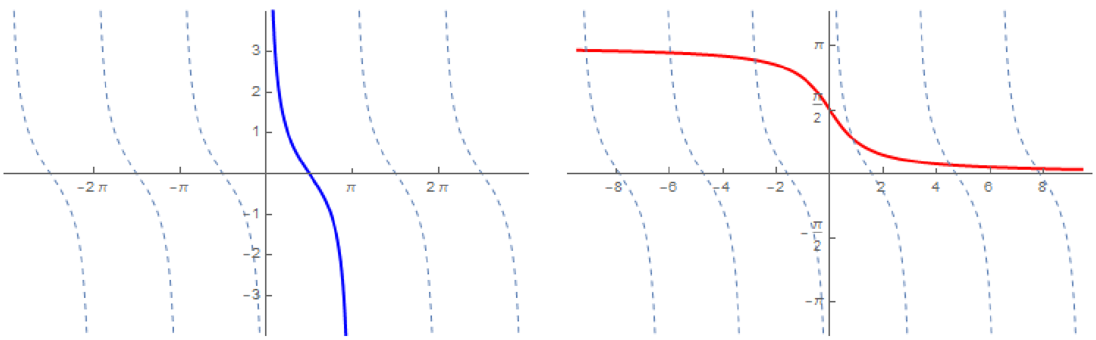
Cosecant Function:



Secant Function:

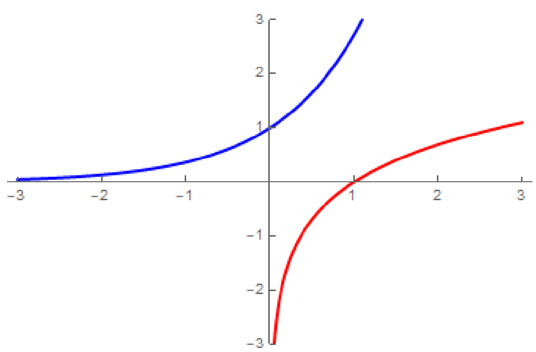


Cotangent Function:



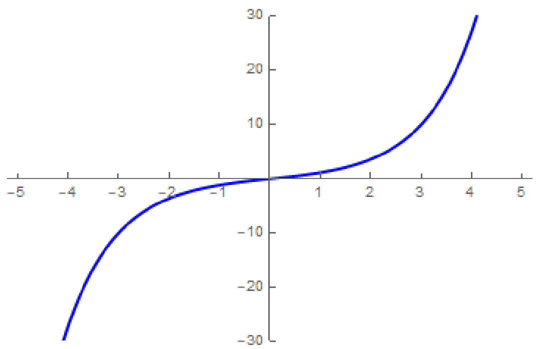
6.1.2 Exponential Functions

Exponential Function with base e:

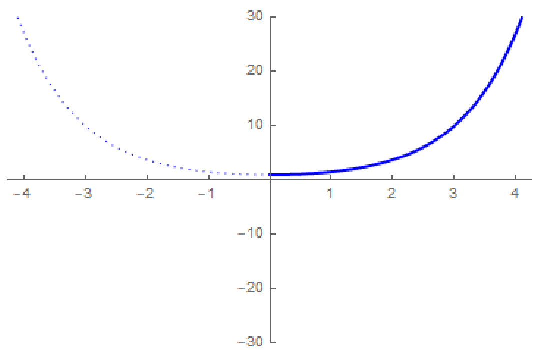


6.1.3 Hyperbolic Functions

Hyperbolic Sine:



Hyperbolic Cosine:



Hyperbolic Tangent: