1 Integrating Rational Functions

We can use algebraic techniques to integrate ratio of polynomials (also called rational functions),

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}.$$

1.1 Partial Fraction Decomposition

If $m > n$, that is the degree of the denominator is larger than the degree of the numerator, then we can use partial fractions to write a rational function as the sum of simpler rational functions. We care about three main types of terms in the denominator,

1. *Distinct Degree 1 Factors:* If the denominator is a product of distinct linear factors, then use partial fractions of the form
$$\frac{A}{ax-b}.$$
2. *Repeated Degree 1 Factors:* If the denominator contains a term of the form $(ax-b)^2$, use partial fractions of the form
$$\frac{A}{(ax-b)^2} + \frac{B}{ax-b}.$$
3. *Irreducible Degree 2 Factors:* If the denominator contains a unfactorable term of the form $ax^2 + bx + c$ use partial fractions of the form
$$\frac{Ax+B}{ax^2+bx+c}.$$

Example 1. The partial fraction decomposition implies there exists constants such that

$$\frac{10x-2x^2}{(x-1)^2(x+3)} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x+3}.$$

Example 2. The partial fraction decomposition implies there exists constants such that

$$\frac{x}{(x-1)(x-3)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-3)^2} + \frac{C}{x-3} + \frac{Dx+E}{x^2+1}.$$

1.2 Example Problems

1.2.1 Partial Fraction Decompositions Problems

Heaviside cover-up Method: We will use limits to find the constants in the partial fraction decomposition. If the denominator only contains degree 1 or repeated degree 1 factors, then we can take limits at the singularities to recover the coefficients belonging to the highest power terms. Taking limits to $\pm\infty$ can then be used to recover the coefficients belonging to lower power terms.

The limits in the first 4 example problems are done in full detail to demonstrate the reasoning behind the method. Problem 5 explains how one will apply these methods to essentially do the partial fractions step in our head. The cover-up method works best if there at most one repeated degree 1 factor and no irreducible degree 2 factors.

Remark: The other way to solve for the coefficients involves multiplying by the least common denominator and equating coefficients. We will not demonstrate this method in these notes.

Problem 1. ($\star$) Decompose $\frac{x-7}{(x-1)(x+2)}$ into partial fractions.

Solution 1.

Step 1: The partial fraction decomposition is of the form

$$\frac{x-7}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

for some yet to be determined coefficients A and B.

Step 2: We now find the coefficients.

A: To find A, we study the asymptotics as $(x-1) \to 0$. We multiply both sides by $(x-1)$ and notice

$$A = \lim_{x\to 1}\left(A + \frac{B(x-1)}{x+2}\right) = \lim_{x\to 1}\frac{x-7}{(x+2)} = -2.$$

B: To find B, we study the asymptotics as $(x+2) \to 0$. We multiply both sides by $(x+2)$ and notice

$$B = \lim_{x\to -2}\left(\frac{A(x+2)}{x-1} + B\right) = \lim_{x\to -2}\frac{x-7}{(x-1)} = 3.$$

Step 3: Therefore, we have

$$\frac{x-7}{(x-1)(x+2)} = -\frac{2}{x-1} + \frac{3}{x+2}.$$

Problem 2. ($\star\star$) Decompose $\frac{4x^2}{(x-1)(x-2)^2}$ into partial fractions.

Solution 2.

Step 1: The partial fraction decomposition is of the form

$$\frac{4x^2}{(x-1)(x-2)^2} = \frac{A}{x-1} + \frac{B}{(x-2)^2} + \frac{C}{x-2}$$

for some yet to be determined coefficients A, B, and C.

Step 2: We now find the coefficients.

A: To find A, we study the asymptotics as $(x-1) \to 0$. We multiply both sides by $(x-1)$ and notice

$$A = \lim_{x\to 1}\left(A + \frac{B(x-1)}{(x-2)^2} + \frac{C(x-1)}{x-2}\right) = \lim_{x\to 1}\frac{4x^2}{(x-2)^2} = 4.$$

B: To find B, we study the asymptotics as $(x-2)^2 \to 0$. We multiply both sides by $(x-2)^2$ and notice

$$B = \lim_{x\to 2}\left(\frac{A(x-2)^2}{x-1} + B + \frac{C(x-2)^2}{x-2}\right) = \lim_{x\to 2}\frac{4x^2}{x-1} = 16.$$

C: To find C, we study the asymptotics as $x \to \infty$. We multiply both sides by $(x-2)$ and notice

$$A + C = \lim_{x\to\infty}\left(\frac{A(x-2)}{x-1} + \frac{B}{(x-2)} + C\right) = \lim_{x\to\infty}\frac{4x^2}{(x-1)(x-2)} = 4,$$

and since $A = 4$ we have

$$C = 4 - A = 0.$$

Step 3: Therefore, we have

$$\frac{4x^2}{(x-1)(x-2)^2} = \frac{4}{x-1} + \frac{16}{(x-2)^2}.$$

Problem 3. (⋆⋆) Decompose $\frac{9x+25}{(x+3)^2}$ into partial fractions.

Solution 3.

Step 1: The partial fraction decomposition is of the form

$$\frac{9x+25}{(x+3)^2} = \frac{A}{(x+3)^2} + \frac{B}{(x+3)}$$

for some yet to be determined coefficients A and B.

Step 2: We now find the coefficients.

A: To find A, we study the asymptotics as $(x+3)^2 \to 0$. We multiply both sides by $(x+3)^2$ and notice

$$A = \lim_{x\to -3}\left(A + \frac{B(x+3)^2}{(x+3)}\right) = \lim_{x\to -3}\frac{(9x+25)(x+3)^2}{(x+3)^2} = -2.$$

B: To find B, we study the asymptotics as $x \to \infty$. We multiply both sides by $(x+3)$ and notice

$$B = \lim_{x\to\infty}\left(\frac{A(x+3)}{(x+3)^2} + \frac{B(x+3)}{(x+3)}\right) = \lim_{x\to\infty}\frac{(9x+25)(x+3)}{(x+3)^2} = 9.$$

Step 3: Therefore, we have

$$\frac{9x+25}{(x+3)^2} = -\frac{2}{(x+3)^2} + \frac{9}{(x+3)}.$$

Problem 4. (⋆⋆) Decompose $\frac{x+7}{x^2(x+2)}$ into partial fractions.

Solution 4.

Step 1: The partial fraction decomposition is of the form

$$\frac{x+7}{x^2(x+2)} = \frac{A}{(x+2)} + \frac{B}{x^2} + \frac{C}{x}$$

for some yet to be determined coefficients A, B and C.

Step 2: We now find the coefficients (skipping some details).

A: To find A, we study the asymptotics as $(x+2) \to 0$. We multiply both sides by $(x+2)$ and notice

$$A = \lim_{x\to -2}\frac{x+7}{x^2} = \frac{5}{4}.$$

B: To find B, we study the asymptotics as $x^2 \to 0$. We multiply both sides by x^2 and notice

$$B = \lim_{x\to 0}\frac{x+7}{x+2} = \frac{7}{2}.$$

C: To find C, we study the asymptotics as $x \to \infty$. We multiply both sides by x and notice

$$C + A = \lim_{x\to\infty} \left(C + A \cdot \frac{x}{x+2} \right) = \lim_{x\to\infty} \frac{x+7}{x(x+2)} = 0 \implies C = -A = -\frac{5}{4}.$$

Step 3: Therefore, we have

$$\frac{x+7}{x^2(x+2)} = \frac{5}{4(x+2)} + \frac{7}{2x^2} - \frac{5}{4x}.$$

Problem 5. ($\star$) Decompose the following functions into partial fractions

1. $$\frac{7x-6}{(x-2)(x+3)}$$
2. $$\frac{x^2+x-1}{x(2x-1)(x+3)}$$
3. $$\frac{10x^2-2x}{(x-1)^2(x+3)}$$

Solution 5. We can do the computations in the above examples in our head. This method is called the Heaviside cover-up method. We demonstrate this with a few examples:

1. Since
$$\left.\frac{7x-6}{x+3}\right|_{x=2} = \frac{8}{5}, \qquad \left.\frac{7x-6}{x-2}\right|_{x=-3} = \frac{27}{5}$$
we can conclude
$$\frac{7x-6}{(x-2)(x+3)} = \frac{8}{5} \cdot \frac{1}{x-2} + \frac{27}{5} \cdot \frac{1}{x+3}.$$
2. Since
$$\left.\frac{x^2+x-1}{(2x-1)(x+3)}\right|_{x=0} = \frac{1}{3}, \quad \left.\frac{x^2+x-1}{x(x+3)}\right|_{x=\frac{1}{2}} = -\frac{1}{7}, \quad \left.\frac{x^2+x-1}{x(2x-1)}\right|_{x=-3} = \frac{5}{21}$$
we can conclude
$$\frac{x^2+x-1}{x(2x-1)(x+3)} = \frac{1}{3} \cdot \frac{1}{x} - \frac{1}{7} \cdot \frac{1}{2x-1} + \frac{5}{21} \cdot \frac{1}{x+3}.$$
3. We solve for the highest power terms first. Since
$$\left.\frac{10x^2-2x}{x+3}\right|_{x=1} = 2, \qquad \left.\frac{10x^2-2x}{(x-1)^2}\right|_{x=-3} = 6$$
we can conclude
$$\frac{10x^2-2x}{(x-1)^2(x+3)} = \frac{2}{(x-1)^2} + \frac{B}{x-1} + \frac{6}{x+3}.$$
Taking the limit as $x \to \infty$ (or looking at the order $\frac{1}{x}$ terms), we can conclude
$$10 = B + 6 \implies B = 4.$$
Therefore,
$$\frac{10x^2-2x}{(x-1)^2(x+3)} = \frac{2}{(x-1)^2} + \frac{4}{x-1} + \frac{6}{x+3}.$$

Extra Practice: Redo Problems 1 – 4 using the shortcut described in Problem 5.

1.2.2 Integrating Rational Functions

Strategy: We want to use algebra to simplify our integral to a form that is easier to integrate.

1. If the degree of the numerator is bigger than or equal to the denominator, use long division first to simplify the integral.
2. If there is a term with bigger degree in the denominator, use partial fraction decompositions to split the rational function into more manageable parts.
3. Use linearity of integration to compute the integral.

Problem 1. $(\star)$ Evaluate the integral

$$\int \frac{x^4}{x-1}\,dx.$$

Solution 1. The degree of the numerator is bigger, so we first use long division,

$$\begin{aligned}\int \frac{x^4}{x-1}\,dx &= \int x^3+x^2+x+1+\frac{1}{x-1}\,dx && \text{long division}\\ &= \frac{x^4}{4}+\frac{x^3}{3}+\frac{x^2}{2}+x+\ln|x-1|+C.\end{aligned}$$

Details of the long division step: By polynomial long division,

$$\begin{array}{r|l} & x^3+x^2+x+1 \\ \hline x-1 & x^4 \\ & -x^4+x^3 \\ \hline & x^3 \\ & -x^3+x^2 \\ \hline & x^2 \\ & -x^2+x \\ \hline & x \\ & -x+1 \\ \hline & 1 \end{array}$$

Problem 2. $(\star)$ Evaluate the integral

$$\int \frac{x^2+x+1}{(x+1)^2(x+2)}\,dx.$$

Solution 2. The degree of the denominator is bigger, so we can use partial fractions,

$$\begin{aligned}\int \frac{x^2+x+1}{(x+1)^2(x+2)}\,dx &= \int \frac{1}{(x+1)^2}-\frac{2}{x+1}+\frac{3}{x+2}\,dx && \text{partial fractions}\\ &= -\frac{1}{x+1}-2\ln|x+1|+3\ln|x+2|+C.\end{aligned}$$

Details of the partial fractions step: Since

$$\left.\frac{x^2+x+1}{(x+2)}\right|_{x=-1}=1 \qquad \text{and} \qquad \left.\frac{x^2+x+1}{(x+1)^2}\right|_{x=-2}=3,$$

the cover-up method implies that

$$\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{1}{(x+1)^2} + \frac{B}{x+1} + \frac{3}{x+2}.$$

Taking the limit as $x \to \infty$ and looking at the coefficients of the $\frac{1}{x}$ terms implies that

$$1 = 0 + B + 3 \implies B = -2.$$

Problem 3. ($\star\star$) Evaluate the integral

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx.$$

Solution 3. The degree of the numerator is bigger, so we first use long division followed by partial fractions,

$$\begin{aligned}
\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx &= \int x + 1 + \frac{4x}{x^3 - x^2 - x + 1}\, dx && \text{long division} \\
&= \int x + 1 + \frac{4x}{(x-1)^2(x+1)}\, dx && \text{factoring} \\
&= \int x + 1 + \frac{2}{(x-1)^2} + \frac{1}{x-1} - \frac{1}{x+1}\, dx && \text{partial fractions} \\
&= \frac{x^2}{2} + x - \frac{2}{x-1} + \ln|x-1| - \ln|x+1| + C.
\end{aligned}$$

Details of the long division step: By polynomial long division,

$$\begin{array}{r|rrrrr}
 & & & & x & +1 \\ \hline
x^3 - x^2 - x + 1 & x^4 & & -2x^2 & +4x & +1 \\
 & -x^4 & +x^3 & +x^2 & -x & \\ \hline
 & & x^3 & -x^2 & +3x & +1 \\
 & & -x^3 & +x^2 & +x & -1 \\ \hline
 & & & & 4x &
\end{array}$$

Details of the partial fractions step: Since

$$\left.\frac{4x}{(x+1)}\right|_{x=1} = 2 \qquad \text{and} \qquad \left.\frac{4x}{(x-1)^2}\right|_{x=-1} = -1,$$

the cover-up method implies that

$$\frac{4x}{(x-1)^2(x+1)} = \frac{2}{(x-1)^2} + \frac{B}{x-1} - \frac{1}{x+1}.$$

Taking the limit as $x \to \infty$ and looking at the coefficients of the $\frac{1}{x}$ terms implies that

$$0 = 0 + B - 1 \implies B = 1.$$

Problem 4. ($\star\star$) Evaluate the integral

$$\int \frac{e^{2x}}{e^{2x} + 3e^x + 2}\, dx.$$

Solution 4. We first use a change of variables to simplify our integral then use partial fractions to compute the resulting rational function,

$$\begin{aligned}
\int \frac{e^{2x}}{e^{2x}+3e^x+2}\,dx &= \int \frac{e^x \cdot e^x}{(e^x)^2+3e^x+2}\,dx \\
&= \int \frac{u}{u^2+3u+2}\,du && u = e^x,\ du = e^x dx \\
&= \int \frac{u}{(u+1)(u+2)}\,du && \text{factoring} \\
&= \int -\frac{1}{u+1} + \frac{2}{u+2}\,du && \text{partial fractions} \\
&= -\ln|u+1| + 2\ln|u+2| + C \\
&= -\ln|e^x+1| + 2\ln|e^x+2| + C && u = e^x.
\end{aligned}$$

Details of the partial fractions step: Since

$$\frac{u}{(u+2)}\bigg|_{u=-1} = -1 \qquad \text{and} \qquad \frac{u}{(u+1)}\bigg|_{u=-2} = 2,$$

the cover-up method implies that

$$\frac{u}{(u+1)(u+2)} = -\frac{1}{u+1} + \frac{2}{u+2}.$$

2 Integrating Trigonometric Functions

The following trigonometric identities will be useful when integrating trigonometric functions.

1. Pythagorean Formulas:
$$\sin^2(\theta) + \cos^2(\theta) = 1 \tag{1}$$
$$\tan^2(\theta) + 1 = \sec^2(\theta) \tag{2}$$
2. Half Angle Formulas:
$$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2} \tag{3}$$
$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2} \tag{4}$$
3. Product to Sum Formulas:
$$\cos(\theta)\cos(\varphi) = \frac{1}{2}(\cos(\theta+\varphi) + \cos(\theta-\varphi)) \tag{5}$$
$$\sin(\theta)\sin(\varphi) = \frac{1}{2}(\cos(\theta-\varphi) - \cos(\theta+\varphi)) \tag{6}$$
$$\sin(\theta)\cos(\varphi) = \frac{1}{2}(\sin(\theta+\varphi) + \sin(\theta-\varphi)) \tag{7}$$

2.1 Example Problems

Strategy: Use trigonometric identities to simplify our integral. In most cases, our goal will be to simplify the function until we have exactly one $\sin(x)$ or $\cos(x)$ term appearing. This will put our integral in a form that we can use a substitution to compute.

Problem 1. ($\star$) Evaluate the integral
$$\int \sin(5x)\cos(2x)\,dx.$$

Solution 1. We simplify the integral using the product to sum formulas,
$$\int \sin(5x)\cos(2x)\,dx = \frac{1}{2}\int \sin(7x) + \sin(3x)\,dx = -\frac{1}{14}\cos(7x) - \frac{1}{6}\cos(3x) + C.$$

Problem 2. ($\star\star$) Evaluate the integral
$$\int \cos^4(x)\,dx.$$

Solution 2. Using trigonometric identities to simplify the integral,
$$\begin{aligned}
\int \cos^4(x)\,dx = \int (\cos^2(x))^2\,dx &= \int \left(\frac{1+\cos 2\theta}{2}\right)^2 dx && \text{Half Angle}\\
&= \frac{1}{4}\int 1 + 2\cos(2x) + \cos^2(2x)\,dx\\
&= \frac{1}{4}\int 1 + 2\cos(2x) + \frac{1}{2}\Big(1+\cos(4x)\Big)\,dx && \text{Half Angle}\\
&= \frac{1}{4}\int \frac{3}{2} + 2\cos(2x) + \frac{1}{2}\cos(4x)\,dx\\
&= \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C.
\end{aligned}$$

Problem 3. ($\star\star$) Evaluate the integral

$$\int \cos^3(x)\sin^4(x)\,dx.$$

Solution 3. Using trigonometric identities to simplify the integral,

$$\begin{aligned}
\int \cos^3(x)\sin^4(x)\,dx &= \int (1-\sin^2(x))\sin^4(x)\cos(x)\,dx && \text{Pythagorean Formula}\\
&= \int \sin^4(x)\cos(x)\,dx - \int \sin^6(x)\cos(x)\,dx\\
&= \int u^4\,du - \int u^6\,du && u=\sin(x),\ du=\cos(x)\,dx\\
&= \frac{u^5}{5} - \frac{u^7}{7} + C\\
&= \frac{\sin^5(x)}{5} - \frac{\sin^7(x)}{7} + C. && u=\sin(x)
\end{aligned}$$

Problem 4. ($\star\star$) Evaluate the integral

$$\int \sec(x)\,dx.$$

Solution 4. We will present a derivation using partial fractions,

$$\begin{aligned}
\int \sec(x)\,dx &= \int \frac{\cos(x)}{\cos^2(x)}\,dx && \text{Multiply and Divide by } \cos(x)\\
&= \int \frac{\cos(x)}{1-\sin^2(x)}\,dx && \text{Pythagorean Formula}\\
&= \int \frac{1}{1-u^2}\,du && u=\sin(x),\ du=\cos(x)\,dx\\
&= \int \frac{1}{(1-u)(1+u)}\,du && \text{Factoring}\\
&= \int \frac{1}{2}\cdot\frac{1}{1-u} + \frac{1}{2}\cdot\frac{1}{1+u}\,du && \text{Partial fractions}\\
&= -\frac{1}{2}\ln|1-u| + \frac{1}{2}\ln|1+u| + C\\
&= \frac{1}{2}\ln\left|\frac{1+\sin(x)}{1-\sin(x)}\right| + C. && u=\sin(x)
\end{aligned}$$

Remark: This solution may appear to give us a different answer than Problem 6 in Week 3,

$$\int \sec(x)\,dx = \ln|\sec(x)+\tan(x)| + C.$$

Our answers are equivalent, and we can use algebra to reduce from one to another,

$$\begin{aligned}
\frac{1}{2}\ln\left|\frac{1+\sin(x)}{1-\sin(x)}\right| + C &= \frac{1}{2}\ln\left|\frac{(1+\sin(x))^2}{1-\sin^2(x)}\right| + C && \text{Multiply and Divide by } 1+\sin(x)\\
&= \ln\left|\frac{(1+\sin(x))^2}{\cos^2(x)}\right|^{1/2} + C && \text{Pythagorean Formula}\\
&= \ln\left|\frac{1+\sin(x)}{\cos(x)}\right| + C\\
&= \ln|\sec(x)+\tan(x)| + C.
\end{aligned}$$

Problem 5. ($\star\star$) Evaluate the integral

$$\int \sec^3(x)\,dx.$$

Solution 5. We need to find the integral of a trigonometric function.

Step 1: We can integrate by parts first. From the table

$\pm$	D	I
$+$	$\sec(x)$	$\sec^2(x)$
$-\int$	$\sec(x)\tan(x)$	$\tan(x)$

we see that

$$\begin{aligned}
\int \sec^3(x)\,dx &= \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\,dx \\
&= \sec(x)\tan(x) - \int \sec(x)(\sec^2(x)-1)\,dx && \text{Pythagorean Formula} \\
&= \sec(x)\tan(x) - \int \sec^3(x)\,dx + \int \sec(x)\,dx \\
&= \sec(x)\tan(x) - \int \sec^3(x)\,dx + \ln|\sec(x)+\tan(x)| && \text{Problem 4.}
\end{aligned}$$

Step 2: Rearranging terms and remembering to include the integrating constant, we see that

$$2\int \sec^3(x)\,dx = \sec(x)\tan(x) + \ln|\sec(x)+\tan(x)| + C$$

and therefore,

$$\int \sec^3(x)\,dx = \frac{1}{2}\Big(\sec(x)\tan(x) + \ln|\sec(x)+\tan(x)| + C\Big).$$