1 Indefinite Integrals

Definition 1. The *antiderivative* of $f(x)$ on $[a, b]$ is a function $F(x)$ such that $F'(x) = f(x)$ for all $x \in [a, b]$. The antiderivative is not unique since for any antiderivative $F(x)$ of $f(x)$, we also have that $F(x) + C$ is an antiderivative for all integration constants $C \in \mathbb{R}$. We will use the notation

$$\int f(x)\,dx = F(x) + C$$

called the *indefinite integral* of f to denote the family of all antiderivatives of f.

1.1 Table of Integrals (omitting the integration constant)

Power Functions

$$\int x^a\,dx = \frac{1}{a+1}\cdot x^{a+1}, \quad a \neq -1 \tag{1}$$

$$\int \frac{1}{x}\,dx = \ln|x| \tag{2}$$

Exponential and Logarithms

$$\int e^x\,dx = e^x \tag{3}$$

$$\int \ln(x)\,dx = x\ln(x) - x \tag{4}$$

$$\int a^x\,dx = \frac{a^x}{\ln(a)} \tag{5}$$

$$\int \log_a(x)\,dx = \frac{x\ln(x) - x}{\ln(a)} \tag{6}$$

Trigonometric Functions

$$\int \sin(x)\,dx = -\cos(x) \tag{7}$$

$$\int \cos(x)\,dx = \sin(x) \tag{8}$$

$$\int \tan(x)\,dx = -\ln|\cos(x)| \tag{9}$$

$$\int \cot(x)\,dx = \ln|\sin(x)| \tag{10}$$

$$\int \sec(x)\,dx = \ln|\tan(x) + \sec(x)| \tag{11}$$

$$\int \csc(x)\,dx = -\ln|\cot(x) + \csc(x)| \tag{12}$$

Rational Functions

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) \tag{13}$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) \tag{14}$$

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}(|x|) \tag{15}$$

Hyperbolic Functions

$$\int \sinh(x)\,dx = \cosh(x) \tag{16}$$

$$\int \cosh(x)\,dx = \sinh(x) \tag{17}$$

$$\int \tanh(x)\,dx = \ln|\cosh(x)| \tag{18}$$

Remark: Formulas (1), (2), (3), (7), (8), (13) are the important ones to memorize for this course.

1.2 Basic Property of Indefinite Integrals

For constants $a, b \in \mathbb{R}$ and integrable functions f and g, we have

Linearity:

$$\int (af(x) + bg(x))\,dx = a\int f(x)\,dx + b\int g(x)\,dx.$$

We will introduce more basic properties in the coming weeks.

1.3 Example Problems

1.3.1 Finding Indefinite Integrals

Problem 1. ($\star$) Find the indefinite integral

$$\int e^{-7x}\, dx.$$

Solution 1. It is easy to check that $-\frac{1}{7} \cdot e^{-7x}$ is an antiderivative of e^{-7x}. Therefore,

$$\int e^{-7x}\, dx = -\frac{1}{7} \cdot e^{-7x} + C.$$

Problem 2. ($\star\star$) Find the indefinite integral

$$\int \frac{x^3+8}{x}\, dx.$$

Solution 2. Using the linearity property, we have

$$\int \frac{x^3+8}{x}\, dx = \int x^2\, dx + 8\int \frac{1}{x}\, dx = \frac{1}{3}x^3 + 8\ln|x| + C.$$

1.3.2 Checking Antiderivatives

Strategy: To check if a function $F(x)$ is an antiderivative of f, it suffices to just differentiate F and check if $F'(x) = f(x)$.

Problem 3. ($\star\star$) Check that both $-\sin^{-1}(x)$ and $\cos^{-1}(x)$ are antiderivatives of

$$\frac{-1}{\sqrt{1-x^2}} \text{ on } (-1,1).$$

Show that

$$\cos^{-1}(x) + \sin^{-1}(x) = \frac{\pi}{2}.$$

Solution 3. From the table of derivatives, we have

$$\frac{d}{dx}(-\sin^{-1}(x)) = \frac{-1}{\sqrt{1-x^2}} \text{ and } \frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}.$$

so both $-\sin^{-1}(x)$ and $\cos^{-1}(x)$ are antiderivatives of $\frac{-1}{\sqrt{1-x^2}}$ on its domain. We know that all antiderivatives on an interval differ by an integration constant, that is $\cos^{-1}(x) = -\sin^{-1}(x) + C$ for $x \in (-1,1)$. To find this integration constant, we can evaluate our functions at 0,

$$\cos^{-1}(0) = \frac{\pi}{2} \text{ and } -\sin^{-1}(0) = 0 \implies \cos^{-1}(0) = -\sin^{-1}(0) + C \implies C = \frac{\pi}{2}.$$

Since the difference of any two antiderivatives is a constant, we conclude that the two antiderivatives must differ by $\frac{\pi}{2}$ for all x in the domain,

$$\cos^{-1}(x) = -\sin^{-1}(x) + \frac{\pi}{2} \implies \cos^{-1}(x) + \sin^{-1}(x) = \frac{\pi}{2}.$$

2 The Fundamental Theorem of Calculus

The fundamental theorem of calculus links the notions of differentiation and integration. Consider the area function of f from a to x,

$$A(x) = \int_a^x f(t)\,dt.$$

One would expect that the rate of change of the area at the point $x = b$ should be proportional to the height of the function $f(b)$, that is $A'(b) = f(b)$. We can think of integration as the 'inverse of differentiation'. This notion is made precise with the fundamental theorem of calculus.

Theorem 1 (Fundamental Theorem of Calculus)**.** *Suppose f is a continuous function on $[a, b]$. Then*

1. *$A(x)$ is an antiderivative of f on (a, b). In particular, for all $x \in (a, b)$,*

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

2. *If F is an antiderivative of f on $[a, b]$, then*

$$\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = F(x)\Big|_{x=a}^{x=b} = F(b) - F(a).$$

Proof. We start by proving the first part of the fundamental theorem of calculus.

Proof of Part 1: Let $A(x)$ be the area function of f from a to x

$$A(x) = \int_a^x f(t)\,dt.$$

We can compute the derivative using the limit definition. Let $x \in (a, b)$, we have

$$A'(x) = \lim_{h\to 0}\frac{A(x+h) - A(x)}{h} = \lim_{h\to 0}\frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h} = \lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,dt}{h}.$$

Let $m(h) = \min_{t\in[x-|h|,x+|h|]} f(t)$ and $M(h) = \max_{t\in[x-|h|,x+|h|]} f(t)$ be the corresponding maximum and minimum of $f(t)$ on the interval $[x - |h|, x + |h|]$. These quantities exist by the Extreme Value Theorem because f is continuous and the interval is closed. We can now apply the squeeze theorem to compute our limit,

$$\begin{aligned} m(h) \le f(t) \le M(h) &\Rightarrow \frac{\int_x^{x+h} m(h)\,dt}{h} \le \frac{\int_x^{x+h} f(t)\,dt}{h} \le \frac{\int_x^{x+h} M(h)\,dt}{h} \qquad \text{also holds for } h < 0 \\ &\Rightarrow \frac{m(h)(x+h-x)}{h} \le \frac{\int_x^{x+h} f(t)\,dt}{h} \le \frac{M(h)(x+h-x)}{h} \\ &\Rightarrow m(h) \le \frac{\int_x^{x+h} f(t)\,dt}{h} \le M(h). \end{aligned}$$

Notice that by continuity, we have $\lim_{h\to 0} m(h) = f(x)$ and $\lim_{h\to 0} M(h) = f(x)$ so

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = \lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,dt}{h} = f(x).$$

Remark: We have shown that $F(x)$ is differentiable for all $x \in (a, b)$ and therefore continuous on (a, b). It is also possible to show that $F(x)$ is continuous at the endpoints.

Proof of Part 2: From Part 1, we know that $A(x)$ is an antiderivative of $f(x)$ on $[a, b]$. If $F(x)$ is another antiderivative of f, then we know that $A(x) = F(x) + C$ for some integration constant C and all $x \in [a, b]$. To find the integration constant, we evaluate our function at $x = a$ and conclude

$$A(a) = F(a) + C \implies 0 = F(a) + C \implies C = -F(a).$$

Therefore,

$$\int_a^b f(t)\,dt = A(b) = F(b) + C = F(b) - F(a).$$

$\square$

2.1 Example Problems

2.1.1 Application of the Fundamental Theorem of Calculus Part 1

Problem 1. ($\star$) Let $g(x) = \int_{-5}^{x} e^{-t^2}\,dt$. Find $g'(x)$.

Solution 1. By the fundamental theorem of calculus, we have

$$g'(x) = e^{-x^2}.$$

Problem 2. ($\star\star$) Let $g(x) = \int_0^{\ln(x)} e^{-t}\,dt$. Find $g'(x)$.

Solution 2. Let $h(x) = \int_0^x e^{-t}\,dt$. Notice that $g(x) = h(\ln(x))$. Therefore, by the fundamental theorem of calculus and the chain rule, we have

$$g'(x) = h'(\ln(x)) \cdot \frac{d}{dx}\ln(x) = e^{-\ln(x)} \cdot \frac{1}{x} = e^{\ln(x^{-1})} \cdot \frac{1}{x} = \frac{1}{x^2}.$$

We used the fact that $h'(x) = \frac{d}{dx}\int_0^x e^{-t}\,dt = e^{-x}$ by the fundamental theorem of calculus.

Remark: If we used the second part of the fundamental theorem of calculus, then

$$g(x) = \int_0^{\ln(x)} e^{-t}\,dt = -e^{-t}\Big|_{t=0}^{t=\ln(x)} = -e^{-\ln(x)} + 1 = e^{\ln(x^{-1})} + 1 = -\frac{1}{x} + 1.$$

Differentiating this, we see

$$g'(x) = \frac{d}{dx}\Big(-\frac{1}{x} + 1\Big) = \frac{1}{x^2}.$$

Problem 3. ($\star\star$) Let $g(x) = \int_{-x^2}^{4x} \sin^2(t)\,dt$. Find $g'(x)$.

Solution 3. We start by splitting the region of integration

$$g(x) = \int_{-x^2}^{4x} \sin^2(t)\,dt = \int_{-x^2}^{0} \sin(t)\,dt + \int_0^{4x} \sin^2(t)\,dt = -\int_0^{-x^2} \sin^2(t)\,dt + \int_0^{4x} \sin^2(t)\,dt.$$

By the fundamental theorem of calculus and the chain rule, we have

$$\begin{aligned}
g'(x) &= -\frac{d}{dx}\int_0^{-x^2} \sin^2(t)\,dt + \frac{d}{dx}\int_0^{4x} \sin^2(t)\,dt && \text{linearity} \\
&= -\sin^2(-x^2) \cdot \frac{d}{dx}(-x^2) + \sin^2(4x) \cdot \frac{d}{dx}4x && \text{fundamental theorem \& chain rule} \\
&= 2x\sin^2(-x^2) + 4\sin^2(4x).
\end{aligned}$$

The reasoning for the chain rule is the same as the previous problem.

2.1.2 Application of the Fundamental Theorem of Calculus Part 2

Problem 4. ($\star$) Find the area under the curve of e^{-2x} on the interval $[0, 9]$.

Solution 4. If suffices to compute the following definite integral

$$\int_0^9 e^{-2x}\,dx.$$

It is easy to check that $F(x) = -\frac{1}{2}e^{-2x}$ is an antiderivative of e^{-2x}. Therefore, by the fundamental theorem of calculus,

$$\int_0^9 e^{-2x}\,dx = -\frac{1}{2}e^{-2x}\Big|_{x=0}^{x=9} = -\frac{1}{2}e^{-2\cdot 9} + \frac{1}{2}e^{-2\cdot 0} = \frac{1}{2} - \frac{1}{2}e^{-18}.$$

Remark: We can actually compute the area explicitly using Riemann sums. Recall that geometric series has a closed form given by

$$\sum_{i=0}^{n-1} ar^i = a\Big(\frac{1-r^n}{1-r}\Big).$$

Using the left endpoint Riemann sum, we have

$$\lim_{n\to\infty}\frac{9}{n}\cdot\sum_{i=1}^{n} e^{-\frac{18(i-1)}{n}} \overset{j=i-1}{=} \lim_{n\to\infty}\frac{9}{n}\cdot\sum_{j=0}^{n-1}\big(e^{-\frac{18}{n}}\big)^j = \lim_{n\to\infty}\frac{9}{n}\cdot\Big(\frac{1-e^{-18}}{1-e^{-\frac{18}{n}}}\Big) = \frac{1}{2} - \frac{1}{2}e^{-18}$$

using L'Hôpital's rule.

Problem 5. ($\star\star$) Compute the limit

$$\lim_{n\to\infty}\frac{3}{n}\Big(\sqrt{1+\frac{3}{n}} + \sqrt{1+2\cdot\frac{3}{n}} + \sqrt{1+3\cdot\frac{3}{n}} + \cdots + \sqrt{1+n\cdot\frac{3}{n}}\Big).$$

Solution 5. The summation appears to be a right Riemann sum. Recall that the right Riemann sum is given by

$$\lim_{n\to\infty}\sum_{i=1}^{n} f\big(a + i\cdot\Delta x\big)\cdot\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f\Big(a + i\cdot\frac{b-a}{n}\Big)\cdot\frac{b-a}{n}.$$

If we choose $a = 1$, $b = 4$ and $f(x) = \sqrt{x}$, then by the definition of the definite integral, we have

$$\int_1^4 \sqrt{x}\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{1+i\Delta x}\cdot\Delta x = \lim_{n\to\infty}\frac{3}{n}\Big(\sqrt{1+\frac{3}{n}} + \sqrt{1+2\cdot\frac{3}{n}} + \cdots + \sqrt{1+n\cdot\frac{3}{n}}\Big).$$

Therefore, using the fundamental theorem of calculus, the infinite sum is equal to

$$\int_1^4 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_{x=1}^{x=4} = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}.$$

Remark: Notice that if we were to choose $a = 0$, $b = 3$ and $f(x) = \sqrt{1+x}$, then we also have

$$\int_0^3 \sqrt{1+x}\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{1+i\Delta x}\cdot\Delta x = \lim_{n\to\infty}\frac{3}{n}\Big(\sqrt{1+\frac{3}{n}} + \sqrt{1+2\cdot\frac{3}{n}} + \cdots + \sqrt{1+n\cdot\frac{3}{n}}\Big).$$

Therefore, using the Fundamental theorem of calculus to compute this infinite sum, we have

$$\int_0^3 \sqrt{1+x}\,dx = \frac{2}{3}(1+x)^{3/2}\Big|_{x=0}^{x=3} = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}.$$

Once we learn integration by substitution, we will see why these two integrals are equal.

3 Average Value of a Function

The average value of a function f on the interval $[a, b]$ is given by

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

We can interpret f_{ave} as the number such that half the area of the curve lies above it, and half the area lies below it. In other words, f_{ave} is the number such that

$$\int_a^b (f(x) - f_{ave})\,dx = 0.$$

Example: The average of $f(x) = x^2$ and $g(x) = x + 1$ on the interval $[-2, 2]$ is displayed below:

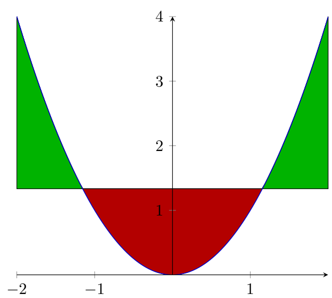

Figure 1: $f_{ave} = \frac{1}{4}\int_{-2}^{2} x^2\,dx = \frac{4}{3}$

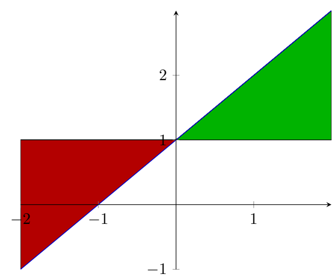

Figure 2: $g_{ave} = \frac{1}{4}\int_{-2}^{2} x + 1\,dx = 1$

3.1 More Properties of Definite Integrals

1. The following theorem says that a continuous function attains its average value:

   Theorem 2 (The Mean Value Theorem for Integrals)**.** *If f is continuous on $[a, b]$, then there exists a number $c \in [a, b]$ such that $f(c) = f_{ave}$. That is, there exists a $c \in [a, b]$ such that*

   $$\int_a^b f(x)\,dx = f(c)(b-a).$$

2. The following result says that if f is continuous and non-negative, then a vanishing integral implies the function vanishes:

   Corollary 1 (Vanishing Theorem)**.** *Suppose $f(x)$ is a continuous function on $[a, b]$. If $f(x) \geq 0$ for all $x \in [a, b]$ and*

   $$\int_a^b f(x)\,dx = 0,$$

   then $f(x) = 0$ for all $x \in [a, b]$.

3. We can "move" the average integral inside of a function that is convex:

   Theorem 3 (Jensen's Inequality)**.** *If $f(x)$ is a continuous function on $[a, b]$ and $g''(x) \geq 0$ for all x in the range of f, then $g(f_{ave}) \leq (g \circ f)_{ave}$. That is,*

   $$g\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right) \leq \frac{1}{b-a}\int_a^b g(f(x))\,dx.$$

3.2 Example Problems

Problem 1. ($\star\star\star$) Prove the Mean Value Theorem for Integration by applying the Mean Value Theorem to the function $F(x) = \int_a^x f(t)\,dt$.

Solution 1. By the fundamental theorem of calculus, $F(x)$ is continuous on $[a, b]$ and $F(x)$ is differentiable on (a, b). Therefore, by the mean value theorem there exists a $c \in (a, b)$ such that

$$\frac{F(b) - F(a)}{b - a} = F'(c) = f(c).$$

Since $F(b) - F(a) = \int_a^b f(x)\,dx - \int_a^a f(x)\,dx = \int_a^b f(x)\,dx$, there exists a $c \in (a, b)$ such that

$$\frac{\int_a^b f(x)\,dx}{b - a} = f(c) \implies \int_a^b f(x)\,dx = f(c)(b - a).$$

Problem 2. ($\star\star\star$) Prove the vanishing theorem. That is, suppose that $f(x)$ is a continuous function on $[a, b]$ such that $f(x) \geq 0$ for all $x \in [a, b]$ and

$$\int_a^b f(x)\,dx = 0.$$

Prove that $f(x) = 0$ for all $x \in [a, b]$.

Solution 2. On the contrary, suppose that $f(x^*) > 0$ for some point $x^* \in [a, b]$. Then by continuity, we must also have that $f(x) > 0$ on some interval $[k, \ell] \subset [a, b]$. By the mean value theorem of integration, there exists a $c \in [k, \ell]$ such that

$$\int_k^\ell f(x)\,dx = f(c)(\ell - k).$$

Since we also have that $f(x) > 0$ for all $x \in [k, \ell]$, we must have $f(c) > 0$, which implies that

$$\int_k^\ell f(x)\,dx = f(c)(\ell - k) > 0.$$

Since $f(x) \geq 0$, by the monotonicity of integration, the conclusion above implies

$$\int_a^b f(x)\,dx \geq \int_k^\ell f(x)\,dx > 0,$$

which contradicts the fact that $\int_a^b f(x)\,dx = 0$. Therefore, we must have that $f(x) = 0$ for all $x \in [a, b]$.

Problem 3. ($\star\star\star$) Prove Jensen's inequality. That is, suppose that $f(x)$ is a continuous function on $[a, b]$ and $g''(x) \geq 0$ for all x. Prove that

$$g\left(\frac{1}{b - a}\int_a^b f(x)\,dx\right) \leq \frac{1}{b - a}\int_a^b g(f(x))\,dx.$$

Solution 3. Let $L(x)$ be the linear approximation of g at the point $x = f_{ave}$. Since $g''(x) \geq 0$, we have $g(x)$ lies above its tangent line so,

$$L(x) := g(f_{ave}) + g'(f_{ave})(x - f_{ave}) \leq g(x).$$

Since this holds for all x, we have $L(f(x)) \leq g(f(x))$. Taking the average integral of both sides and using the monotonicity of definite integrals, we can conclude

$$g(f_{ave}) = \frac{1}{b-a}\int_a^b g(f_{ave}) + g'(f_{ave})(f(x) - f_{ave})\,dx \leq \frac{1}{b-a}\int_a^b g(f(x))\,dx.$$

Remark: We only require $g''(x) \geq 0$ on the range of f for the inequality to hold. This is because f_{ave} is in the range of f by the mean value theorem of integration, and g is only evaluated at points in the range of f. If $g''(x) > 0$, then the same proof shows $g(f_{ave}) < (g \circ f)_{ave}$ provided that f is not a constant function.

Remark: It is easy to see that the opposite inequality holds if $g''(x) \leq 0$. Suppose that $g''(x) \leq 0$, then $-g''(x) \geq 0$, so Jensen's inequality implies that

$$-g(f_{ave}) \leq (-g \circ f)_{ave} \implies g(f_{ave}) \geq (g \circ f)_{ave}.$$

Problem 4. ($\star\star$) Prove the midpoint Riemann sum is an under approximation of the definite integral if $f(x)$ is a convex function.

Solution 4. We can split the region of integration into n uniform subintervals and show that each approximating rectangle using the midpoint is an under approximation of the integral over the subinterval. Therefore, it suffices to show that the midpoint Riemann sum approximation using one rectangle is an under approximation of the definite integral,

$$(b-a)f\Big(\frac{a+b}{2}\Big) \leq \int_a^b f(x)\,dx.$$

The result is immediate from Jensen's inequality. If $f''(x) \geq 0$ on $[a, b]$, then

$$f\Big(\frac{a+b}{2}\Big) = f\Big(\frac{1}{b-a}\int_a^b x\,dx\Big) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \implies (b-a)f\Big(\frac{a+b}{2}\Big) \leq \int_a^b f(x)\,dx,$$

as required.