1 Convergence Tests

There are three major notations of convergence of infinite series:

1. A series is *convergent* if the limit
$$\sum_{n=1}^{\infty} a_n = \lim_{m\to\infty} \sum_{n=1}^{m} a_n$$
exists and is finite.

2. A series is *absolutely convergent* if the limit
$$\sum_{n=1}^{\infty} |a_n| = \lim_{m\to\infty} \sum_{n=1}^{m} |a_n|$$
exists and is finite.

3. A series is *conditionally convergent* if $\sum_{n=1}^{\infty} a_n$ is convergent, but $\sum_{n=1}^{\infty} |a_n|$ is divergent.

We will introduce several techniques that can be used to determine if a series converges without explicitly computing the limit.

1.1 Summary of Convergence Tests for Non-Positive Series

In general, if we add or subtract many terms, then it will be very hard to determine if the series converges because the partial sums are not monotone.

1.1.1 Alternating Series Tests

If $(a_n)_{n\geq 1}$ is a sequence of positive numbers, then an alternating series is a series of the form
$$\sum_{n=1}^{\infty}(-1)^n a_n \qquad \text{or} \qquad \sum_{n=1}^{\infty}(-1)^{n+1} a_n.$$

If the sequence $(a_n)_{n\geq 1}$ is decreasing and $\lim_{n\to\infty} a_n = 0$, then the alternating series converges. In other words, an alternating series with increments decreasing to 0 converges.

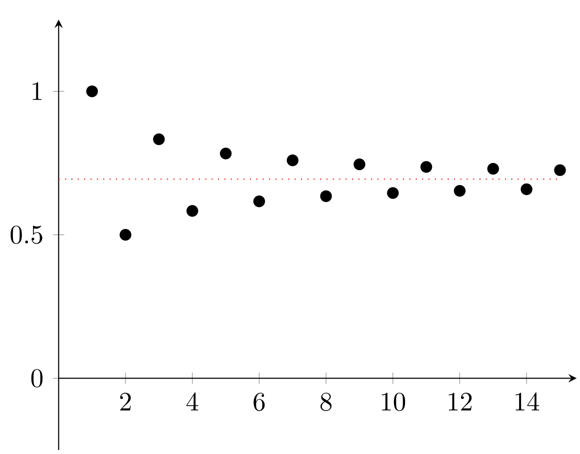

Figure 1: An illustration of the partial sums of $\sum_{n=1}^{m} \frac{(-1)^{n+1}}{n}$. The increments of the sequence is decreasing to 0, so the partial sums oscillate around its limit. In fact, the series converges to $\ln(2)$.

1.2 Summary of Convergence Tests for Positive Series

If all the terms in the sequence are positive (or all negative), then the corresponding series has monotone partial sums. We can use the monotone convergence theorem to develop many tests for convergence.

1.2.1 Comparison Tests

Convergence of a series is dependent on the tails of the series. This is because we can always split the sum into two parts,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N-1} a_n + \sum_{n=N}^{\infty} a_n.$$

The finite sum $\sum_{n=1}^{N-1} a_n$ exists for all N, so we only have to study the behavior of the tails $\sum_{n=N}^{\infty} a_n$. To do comparison tests, we want to find lower bounds or upper bounds on the tail of the series with something we can easily compute. If we show that the upper bound is converges, then the series is also convergent. Likewise, if we show that the lower bound is divergent, then the series is also divergent.

Comparison Test: If $0 \le a_n \le b_n$ for $n \ge N$, then

$$0 \le \sum_{n=N}^{\infty} a_n \le \sum_{n=N}^{\infty} b_n.$$

This means

1. If $\sum_{n=N}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.
2. If $\sum_{n=N}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ also diverges.

Integral Test: Suppose f is continuous, positive and decreasing on $[N, \infty)$. If $a_n = f(n)$ then

$$\int_N^{\infty} f(x)\,dx \le \sum_{n=N}^{\infty} a_n \le f(N) + \int_N^{\infty} f(x)\,dx.$$

This means

1. If $\int_N^{\infty} f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.
2. If $\int_N^{\infty} f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ also diverges.

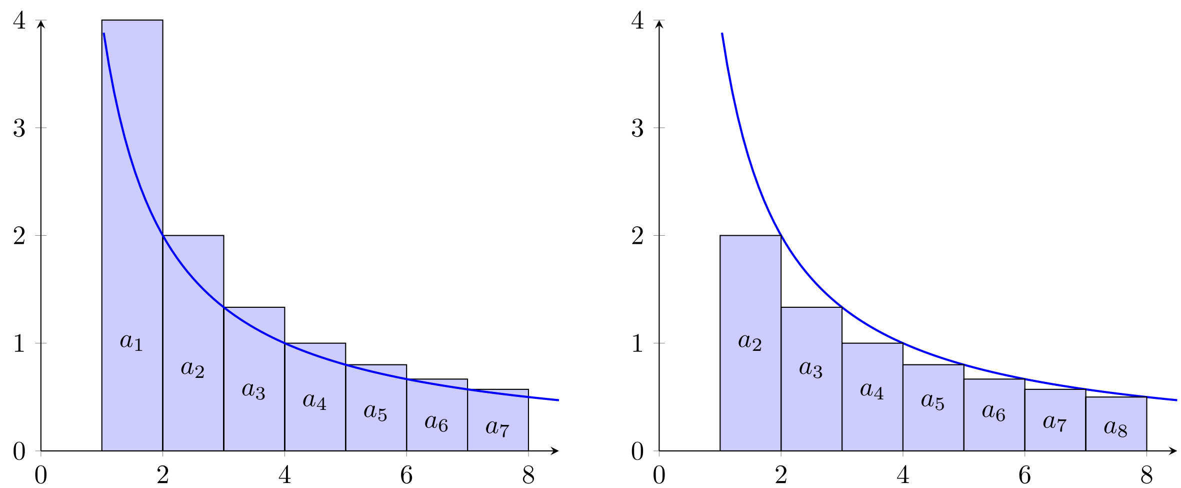

Figure 2: The integral test can be visualized using left and right Riemann sums. The plot on the left shows that $\int_1^{\infty} f(x)\,dx \le \sum_{i=1}^{\infty} a_n$ and the plot on the right shows that $\sum_{n=2}^{\infty} a_n \le \int_1^{\infty} f(x)\,dx$.

1.2.2 Asymptotic Tests

The convergence of a series is determined by its behavior in its tails. This means that we can consider the asymptotic behavior of the sequences to determine its convergence.

Limit Comparison Test: If $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are positive sequences and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \in (0,\infty)$$

then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges (i.e. both series either diverge or converge).

Ratio Test: Suppose the sequence $(a_n)_{n\geq 1}$ satisfies the following limit

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L \neq 1.$$

Using the comparison test on the geometric series implies

1. If $L < 1$ then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
2. If $L > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.

Root Test: Suppose the sequence $(a_n)_{n\geq 1}$ satisfies the following limit

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L \neq 1.$$

Using the comparison test on the geometric series implies

1. If $L < 1$ then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
2. If $L > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.

1.3 Example Problems

The general strategy for testing convergence is as follows:

1. If we have an alternating series, then we always use the alternating series test.
2. If we see $n!$ terms in the summation, then the ratio test will be our best choice. We can use the root test if n^n or a^n appears and the ratio test is hard to compute.
3. If neither case 1 or 2 happen, then we should use the integral test or (limit) comparison test.

Problem 1. ($\star$) Determine whether the following series converges,

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

Solution 1. Since

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \ln(t) = \infty$$

we can conclude that $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ by the integral test.

Problem 2. ($\star$) Determine whether the following series converges,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

Solution 2. We have an alternating series. Since $\frac{1}{n+1} \leq \frac{1}{n}$ for all n, $(\frac{1}{n})_{n\geq 1}$ is decreasing, and $\lim_{n\to\infty} \frac{1}{n} = 0$, so $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by the alternating series test.

Remark: The sequence $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent, since $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges, but $\sum_{n=1}^{\infty} |\frac{(-1)^n}{n}| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Problem 3. ($\star\star$) Determine whether the following series converges,

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2+n}\ln^2(n^3+n^2+n+1)}$$

Solution 3. We will show two approaches to this problem:

Comparison Test: For $n \geq 2$, we have $\sqrt{n^2+n} \geq n$ and $n^3+n^2+n+1 \geq n$. This implies

$$\begin{aligned} & \sqrt{n^2+n}\ln^2(n^3+n^2+n+1) \geq n\ln^2(n) \\ \implies \quad & \frac{1}{\sqrt{n^2+n}\ln^2(n^3+n^2+n+1)} \leq \frac{1}{n\ln^2(n)} \\ \implies & \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2+n}\ln^2(n^3+n^2+n+1)} \leq \sum_{n=2}^{\infty} \frac{1}{n\ln^2(n)} \end{aligned}$$

Since $\int_2^\infty \frac{1}{x\ln^2(x)}\,dx = \lim_{t\to\infty} -\frac{1}{\ln(t)} + \frac{1}{\ln(2)} < \infty$, the integral test implies $\sum_{n=2}^{\infty} \frac{1}{n\ln^2(n)}$ converges. Therefore,

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2+n}\ln^2(n^3+n^2+n+1)} \leq \sum_{n=2}^{\infty} \frac{1}{n\ln^2(n)} < \infty,$$

so $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2+n}\ln^2(n^3+n^2+n+1)}$ converges.

Limit Comparison Test: Asymptotically, we have

$$\frac{1}{\sqrt{n^2+n}\ln^2(n^3+n^2+n+1)} \sim \frac{1}{n\ln^2(n^3)} = \frac{1}{9n\ln^2(n)}.$$

To see this, we can explicitly compute the limit of the ratio,

$$\lim_{x\to\infty} \frac{9x\ln^2(x)}{\sqrt{x^2+x}\ln^2(x^3+x^2+x+1)} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}} \cdot \frac{\ln^2(x^3)}{\ln^2(x^3+x^2+x+1)} = 1,$$

if we use L'Hôpital's rule on the natural log terms. Since

$$\int_2^\infty \frac{1}{9x\ln^2(x)}\,dx = -\frac{1}{9\ln(x)}\Big|_{x=2}^{x=\infty} = \lim_{t\to\infty} -\frac{1}{9\ln(t)} + \frac{1}{9\ln(2)} < \infty,$$

the integral test implies $\sum_{n=2}^{\infty} \frac{1}{9n\ln^2(n)}$ converges. By the limit comparison test, we can conclude that both $\sum_{n=2}^{\infty} \frac{1}{9n\ln^2(n)}$ and $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2+n}\ln^2(n^3+n^2+n+1)}$ converge.

Problem 4. ($\star$) Determine whether the following series converges,

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}.$$

Solution 4. Since $n!$ appears in the summation we will use the ratio test. Notice that

$$\lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1,$$

so the sum converges by the ratio test.

Problem 5. ($\star$) Determine whether the following series converges,

$$\sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^n}.$$

Solution 5. Since n^n appears in the summation we will use the root test. Notice that

$$\lim_{n\to\infty} \Big(\frac{(2n+1)^n}{n^n}\Big)^{1/n} = \lim_{n\to\infty} \frac{2n+1}{n} = 2 > 1,$$

so the sum diverges by the root test.

Problem 6. ($\star\star$) Determine whether the following series converges,

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}.$$

Solution 6. Since $n!$ appears in the summation we will use the ratio test. Notice that

$$\lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \frac{1}{(1+n^{-1})^n} = \frac{1}{e} < 1,$$

so the sum converges by the ratio test.

Problem 7. ($\star\star$) Find the interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{2^{n+1}}$$

Solution 7. We need to find all x values such that the series converges.

Finding the Open Interval of Convergence: By the ratio test, we have

$$\lim_{n\to\infty} \left| \frac{(n+1)(x+2)^{n+1}}{2^{n+2}} \cdot \frac{2^{n+1}}{n(x+2)^n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n} \cdot \frac{(x+2)}{2} \right| = \left| \frac{(x+2)}{2} \right|.$$

For the series to converge, we require $|\frac{(x+2)}{2}| < 1$, which implies that $|x+2| < 2 \implies x \in (-4, 0)$.

Checking the Endpoints: The ratio test also implies the series diverges if $|\frac{(x+2)}{2}| > 1$, so we only have to check if the series converges at the endpoints of the interval $(-4, 0)$.

1. If $x = -4$, then
$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{2^{n+1}}\bigg|_{x=-4} = \sum_{n=0} \frac{n(-2)^n}{2^{n+1}} = \sum_{n=0} \frac{n(-1)^n}{2}$$
which diverges since $\lim_{n\to\infty} \frac{n(-1)^n}{2}$ does not exist.
2. If $x = 0$, then
$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{2^{n+1}}\bigg|_{x=0} = \sum_{n=0} \frac{n2^n}{2^{n+1}} = \sum_{n=0} \frac{n}{2}$$
which diverges since $\lim_{n\to\infty} \frac{n}{2}$ does not exist.

Therefore, the interval of convergence is $(-4, 0)$.

1.3.1 Proofs of Convergence Tests

Problem 1. ($\star\star$) Suppose $f(x)$ is positive and decreasing on $[N, \infty)$. If $a_n = f(n)$ show that
$$\int_N^{\infty} f(x)\,dx \le \sum_{n=N}^{\infty} a_n \le f(N) + \int_N^{\infty} f(x)\,dx.$$

Solution 1. We will show each inequality separately using Riemann sums (see Figure 2).

(a) Since $f(x)$ is decreasing on $[N, \infty)$, the left Riemann sum using subintervals of length 1 is an over approximation of $\int_N^\infty f(x)\,dx$,
$$\int_N^{\infty} f(x)\,dx \le \sum_{n=N}^{\infty} f(n) = \sum_{n=N}^{\infty} a_n.$$

(b) Since $f(x)$ is decreasing on $[N, \infty)$, the right Riemann sum using subintervals of length 1 is an over approximation of $\int_N^\infty f(x)\,dx$,
$$\sum_{n=N+1}^{\infty} a_n = \sum_{n=N+1}^{\infty} f(n) \le \int_N^{\infty} f(x)\,dx.$$

Furthermore, since $a_N = f(N)$, we can add $f(N)$ to both sides to conclude that
$$\sum_{n=N}^{\infty} a_n = f(N) + \sum_{n=N+1}^{\infty} f(n) \le f(N) + \int_N^{\infty} f(x)\,dx.$$

Problem 2. ($\star\star\star$) Prove the ratio test. That is, suppose
$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L \ne 1.$$

(a) If $L < 1$, prove that $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(b) If $L > 1$, prove that $\sum_{n=1}^{\infty} a_n$ diverges.

Solution 2.

(a) Suppose that $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$. There exists a N and a $r < 1$ such that

$$\frac{|a_{n+1}|}{|a_n|} < r \text{ for all } n \geq N \implies |a_{n+1}| < |a_n| r \text{ for all } n \geq N.$$

Applying this inequality recursively we see that

$$|a_{N+k}| < |a_{N+k-1}| r < \cdots < |a_N| r^k.$$

Therefore, we can upper bound the series with a geometric series,

$$\sum_{n=N}^{\infty} |a_n| = \sum_{k=0}^{\infty} |a_{N+k}| < \sum_{k=0}^{\infty} |a_N| r^k = \frac{|a_N|}{1-r} < \infty,$$

since $r < 1$. The comparison test implies that $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N-1} |a_n| + \sum_{n=N}^{\infty} |a_n|$ converges.

(b) Suppose that $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1$. There exists a N such that

$$\frac{|a_{n+1}|}{|a_n|} > 1 \text{ for all } n \geq N \implies |a_{n+1}| > |a_n| \text{ for all } n \geq N.$$

Therefore $\lim_{n\to\infty} |a_n| > |a_N| \geq 0$, which implies that $\lim_{n\to\infty} a_n \neq 0$. Therefore, $\sum_{n=1}^{\infty} a_n$ diverges.

Problem 3. $(\star\star\star)$ Prove the root test. That is, suppose

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L \neq 1.$$

(a) If $L < 1$, prove that $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(b) If $L > 1$, prove that $\sum_{n=1}^{\infty} a_n$ diverges.

Solution 3.

(a) Suppose that $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} < 1$. There exists a N and a $r < 1$ such that

$$|a_n|^{\frac{1}{n}} < r \text{ for all } n \geq N \implies |a_n| < r^n \text{ for all } n \geq N.$$

Therefore, we can upper bound the series with a geometric series,

$$\sum_{n=N}^{\infty} |a_n| < \sum_{n=N}^{\infty} r^n \leq \frac{1}{1-r} < \infty,$$

since $r < 1$. The comparison test implies that $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N-1} |a_n| + \sum_{n=N}^{\infty} |a_n|$ converges.

(b) Suppose that $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} > 1$. There exists a N such that

$$|a_n|^{\frac{1}{n}} > 1 \text{ for all } n \geq N \implies |a_n| > 1 \text{ for all } n \geq N.$$

Therefore $\lim_{n\to\infty} |a_n| \geq 1 \geq 0$, which implies that $\lim_{n\to\infty} a_n \neq 0$. Therefore, $\sum_{n=1}^{\infty} a_n$ diverges.