Solving The Wave Equation

Consider the wave equation on the whole line
\[
\begin{aligned}
&u_{tt} - c^2 u_{xx} = f(x, t) \quad x \in \mathbb{R}, t > 0, \\
&u(x, 0) = \phi(x) \quad x \in \mathbb{R}, \\
u_t(x, 0) = \psi(x) \quad x \in \mathbb{R}.
\end{aligned}
\]

The particular solution to this PDE is given by
\[
\begin{aligned}
u(x, t) &= \frac{\phi(x + ct) + \phi(x - ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) \, ds + \frac{1}{2c} \int_0^t \int_{x-ct}^{x+c(t-s)} f(y, s) \, dy \, ds.
\end{aligned}
\]

Problems

Wave Equation on \( \mathbb{R} \)

**Problem 1.** Solve the initial value problem
\[
\begin{aligned}
&u_{tt} - 4u_{xx} = 0 \quad x \in \mathbb{R}, t > 0, \\
&u(x, 0) = \tanh(x) \quad x \in \mathbb{R}, \\
u_t(x, 0) = \arctan(x) \quad x \in \mathbb{R}.
\end{aligned}
\]

**Solution 1.** By D’Alembert’s formula, the particular solution to this IVP is given by
\[
\begin{aligned}
u(x, t) &= \frac{\tanh(x + 2t) + \tanh(x - 2t)}{2} + \frac{1}{4} \int_{x-2t}^{x+2t} \arctan(y) \, dy.
\end{aligned}
\]

The integral term can be computed using integration by parts,
\[
\frac{1}{4} \int_{x-2t}^{x+2t} \arctan(y) \, dy
= \frac{1}{4} \left( y \arctan(y) - \frac{1}{2} \ln \left| 1 + y^2 \right| \right)_{y=x-2t}^{y=x+2t}
= \frac{1}{4} \left( (x + 2t) \arctan(x + 2t) - (x - 2t) \arctan(x - 2t) - \frac{1}{2} \ln(1 + (x + 2t)^2) + \frac{1}{2} \ln(1 + (x - 2t)^2) \right).
\]

**Problem 2.** Solve the following initial value problems

1. \[
\begin{aligned}
&u_{tt} - 4u_{xx} = 0 \quad x \in \mathbb{R}, t > 0, \\
&u(x, 0) = g(x) \quad x \in \mathbb{R}, \\
u_t(x, 0) = h(x) \quad x \in \mathbb{R}
\end{aligned}
\]

with \( g(x) = \begin{cases} 0 & |x| \geq 1, \\ x^2 - x^4 & |x| < 1, \end{cases} \quad h(x) = 0. \)

2. \[
\begin{aligned}
&u_{tt} - 4u_{xx} = 0 \quad x \in \mathbb{R}, t > 0, \\
&u(x, 0) = g(x) \quad x \in \mathbb{R}, \\
u_t(x, 0) = h(x) \quad x \in \mathbb{R}
\end{aligned}
\]

with \( g(x) = 0, \quad h(x) = \begin{cases} 0 & |x| \geq 1, \\ x^2 - x^4 & |x| < 1. \end{cases} \)
Solution 2.

(1) Since \( h(x) = 0 \), by D’Alembert’s formula, the particular solution to this IVP is given by

\[
u(x, t) = \frac{g(x + 2t) + g(x - 2t)}{2}.
\]

Since \( g(x) \) changes form based on the value of \( |x| \), we can break our solution into 4 cases:

A. \(|x + 2t| \geq 1, |x - 2t| \geq 1\): On this region, \( g(x + 2t) = 0 \) and \( g(x - 2t) = 0 \), so \( u(x, t) = 0 \).

B. \(|x + 2t| < 1, |x - 2t| \geq 1\): On this region, \( g(x + 2t) = (x + 2t)^2 - (x + 2t)^4 \) and \( g(x - 2t) = 0 \), so

\[
u(x, t) = \frac{(x + 2t)^2 - (x + 2t)^4}{2}.
\]

C. \(|x + 2t| \geq 1, |x - 2t| < 1\): On this region, \( g(x + 2t) = 0 \) and \( g(x - 2t) = (x - 2t)^2 - (x - 2t)^4 \), so

\[
u(x, t) = \frac{(x - 2t)^2 - (x - 2t)^4}{2}.
\]

D. \(|x + 2t| < 1, |x - 2t| < 1\): On this region, \( g(x + 2t) = (x + 2t)^2 - (x + 2t)^4 \) and \( g(x - 2t) = (x - 2t)^2 - (x - 2t)^4 \), so

\[
u(x, t) = \frac{(x + 2t)^2 - (x + 2t)^4 + (x - 2t)^2 - (x - 2t)^4}{2}.
\]

Characteristic Lines:
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Description of Picture: The initial condition is supported on the interval $[-1,1]$. The wave propagates right along the lines $x-2t = C \in [-1,1]$ (between the blue characteristic lines) and left along the lines $x+2t = C \in [-1,1]$ (between the green characteristic lines). The behavior on each of the regions can be determined by drawing the domain of dependence at the point $(x,t)$ and seeing if the corners lie in the interval $[-1,1]$. For example, at the point $(-1,0.5)$ the left corner does not lie in $[-1,1]$, while the right corner is in $[-1,1]$, which corresponds to case $B$ above. Similarly, at the point $(0,1.5)$ both corners do not lie in $[-1,1]$, which corresponds to case $A$ above.

(2) Since $g(x) = 0$, by D’Alembert’s formula, the particular solution to this IVP is given by

$$u(x,t) = \frac{1}{4} \int_{x-2t}^{x+2t} h(y) \, dy.$$ 

Since $h(x)$ changes form based on the value of $|x|$, we can break our solution into 5 cases:

A. $x - 2t \leq -1 \leq x + 2t$: On this region, we can split our region of integration into

$$u(x,t) = \frac{1}{4} \int_{x-2t}^{x+2t} h(y) \, dy = \frac{1}{4} \int_{x-2t}^{-1} h(y) \, dy + \frac{1}{4} \int_{-1}^{1} h(y) \, dy + \frac{1}{4} \int_{1}^{x+2t} h(y) \, dy$$

$$= \frac{1}{4} \int_{-1}^{1} y^2 - y^4 \, dy$$

$$= \frac{1}{4} \left( \frac{y^3}{3} - \frac{y^5}{5} \right) \bigg|_{y=-1}^{y=1} = \frac{1}{15}.$$

B. $x - 2t \leq -1 \leq x + 2t \leq 1$: On this region, we can split our region of integration into

$$u(x,t) = \frac{1}{4} \int_{x-2t}^{x+2t} h(y) \, dy = \frac{1}{4} \int_{x-2t}^{-1} h(y) \, dy + \frac{1}{4} \int_{-1}^{1} h(y) \, dy + \frac{1}{4} \int_{1}^{x+2t} h(y) \, dy$$

$$= \frac{1}{4} \int_{-1}^{1} y^2 - y^4 \, dy$$

$$= \frac{1}{4} \left( \frac{y^3}{3} - \frac{y^5}{5} \right) \bigg|_{y=-1}^{y=1} = \frac{(x + 2t)^3}{12} - \frac{(x + 2t)^5}{20} + \frac{1}{30}.$$

C. $-1 \leq x - 2t \leq 1 \leq x + 2t$: On this region, we can split our region of integration into

$$u(x,t) = \frac{1}{4} \int_{x-2t}^{x+2t} h(y) \, dy = \frac{1}{4} \int_{x-2t}^{1} h(y) \, dy + \frac{1}{4} \int_{1}^{x+2t} h(y) \, dy$$

$$= \frac{1}{4} \int_{x-2t}^{1} y^2 - y^4 \, dy$$

$$= \frac{1}{4} \left( \frac{y^3}{3} - \frac{y^5}{5} \right) \bigg|_{y=-2t}^{y=1} = \frac{1}{30} - \frac{(x - 2t)^3}{12} + \frac{(x - 2t)^5}{20}.$$

D. $-1 \leq x - 2t \leq x + 2t \leq 1$: On this region, the integrand is always equal to $h(y) = y^2 - y^4$

$$u(x,t) = \frac{1}{4} \int_{x-2t}^{x+2t} h(y) \, dy = \frac{1}{4} \int_{x-2t}^{x+2t} y^2 - y^4 \, dy$$

$$= \frac{1}{4} \left( \frac{y^3}{3} - \frac{y^5}{5} \right) \bigg|_{y=-2t}^{y=x+2t}$$

$$= \frac{(x + 2t)^3}{12} - \frac{(x + 2t)^5}{20} - \frac{(x - 2t)^3}{12} + \frac{(x - 2t)^5}{20}.$$

E. $x - 2t \geq 1$, or $x + 2t \leq -1$: On this region, the integrand is always equal to $h(y) = 0$, so

$$u(x,t) = 0.$$
Characteristic Lines:

\[ t = 4x \]
\[ t = 3x \]
\[ t = 2x \]
\[ t = x \]
\[ t = -x \]
\[ t = -2x \]
\[ t = -3x \]
\[ t = -4x \]

**Description of Picture:** The initial condition is supported on the interval \([-1, 1]\). The behavior in each of the regions can be determined by drawing the domain of dependence at the point \((x, t)\) and seeing how much of the interval \([-1, 1]\) is contained in the base of the triangle. For example, at \((-1, 0.5)\) the left corner of the base of the triangle is \(<-1\) and the right corner of the base is in \([-1, 1]\), which corresponds to case B above. Similarly, at \((0, 3)\) the left corner of the base of the orange triangle is \(<-1\) and the right corner of the base is in \(>1\), which corresponds to case A above.

**Problem 3.** Solve the initial value problem

\[
\begin{cases}
  u_{tt} - 4u_{xx} = f(x, t) & x \in \mathbb{R}, t > 0, \\
  u(x, 0) = g(x) & x \in \mathbb{R}, \\
  u_t(x, 0) = h(x) & x \in \mathbb{R}
\end{cases}
\]

with

\[
f(x, t) = \begin{cases}
  \sin(x) & 0 < t < \pi \\
  0 & t \geq \pi
\end{cases}, \quad g(x) = 0, \quad h(x) = 0.
\]

**Solution 3.** Since \(g(x) = 0\) and \(h(x) = 0\), by D’Alembert’s formula with the source term the particular solution to this IVP is given by

\[
u(x, t) = \frac{1}{4} \int \int_{\Delta} f(y, s) \, dy \, ds = \frac{1}{4} \int_0^t \int_{x-2(t-s)}^{x+2(t-s)} \sin(y) \mathbb{1}_{[0, \pi]}(s) \, dy \, ds
\]

\[
= \frac{1}{4} \int_0^{\min(t, \pi)} \int_{x-2(t-s)}^{x+2(t-s)} \sin(y) \, dy \, ds.
\]

If you draw the region of integration, we are basically chopping off \(\Delta\) above the line \(t = \pi\) and integrating the remaining trapezoid (or triangle if \(t\) is small enough). We have two cases,
A. $t < \pi$: On this region, we have

$$u(x, t) = \frac{1}{4} \int_0^t \int_{x-2(t-s)}^{x+2(t-s)} \sin(y) \, dy \, ds$$

$$= \frac{1}{4} \int_0^t \left( -\cos(y) \right|_{y=x-2(t-s)}^{y=x+2(t-s)} \, ds$$

$$= \frac{1}{4} \int_0^t -\cos(x + 2(t-s)) + \cos(x - 2(t-s)) \, ds.$$

$$= \frac{1}{8} \left( \sin(x + 2(t-s)) + \sin(x - 2(t-s)) \right) \bigg|_{s=0}^{s=t}$$

$$= \frac{1}{4} \sin(x) - \frac{1}{8} \sin(x + 2t) - \frac{1}{8} \sin(x - 2t).$$

B. $t \geq \pi$: On this region, we can split our region of integration into

$$u(x, t) = \frac{1}{4} \int_0^\pi \int_{x-2(t-s)}^{x+2(t-s)} \sin(y) \, dy \, ds$$

$$= \frac{1}{4} \int_0^\pi \left( -\cos(y) \right|_{y=x-2(t-s)}^{y=x+2(t-s)} \, ds$$

$$= \frac{1}{4} \int_0^\pi -\cos(x + 2(t-s)) + \cos(x - 2(t-s)) \, ds.$$

$$= \frac{1}{8} \left( \sin(x + 2(t-s)) + \sin(x - 2(t-s)) \right) \bigg|_{s=0}^{s=\pi}$$

$$= \frac{1}{8} \sin(x + 2t) + \frac{1}{8} \sin(x - 2t) - \frac{1}{8} \sin(x - 2t)$$

$$= 0.$$

Wave Equation on the Half Line

**Problem 4.** Solve the following PDE

$$\begin{cases}
    u_{tt} = 4u_{xx} & 0 < x < \infty, \quad t > 0 \\
    u(x, 0) = 1 & 0 < x < \infty \\
    u_t(x, 0) = 0 & 0 < x < \infty \\
    u(0, t) = 0 & t > 0
\end{cases}$$

**Solution 4.** We want to solve the wave equation on the half line with Dirichlet boundary conditions. We can use an odd reflection to extend the initial condition,

$$g_{\text{odd}}(x) = \begin{cases}
    1 & x > 0 \\
    0 & x = 0 \\
    -1 & x < 0
\end{cases}, \quad h_{\text{odd}}(x) = 0.$$

The particular solution to the extended PDE is

$$u(x, t) = \frac{g_{\text{odd}}(x + 2t) + g_{\text{odd}}(x - 2t)}{2}.$$

We now examine the cases depending on the sign of $x - 2t$:

1. For $x - 2t > 0$, we have $g_{\text{odd}}(x \pm 2t) = 1$ so the solution is given by

$$u(x, t) = \frac{1 + 1}{2} = 1.$$
2. For $x - 2t < 0$, we have $g_{odd}(x - 2t) = -1$ while $g_{odd}(x + 2t) = 1$, so
\[
u(x, t) = \frac{1 - 1}{2} = 0.
\]

3. When $x - 2t = 0$, we have $g_{odd}(x - 2t) = 0$ so
\[
u(x, t) = \frac{1}{2}.
\]

In summary, for $x > 0$ and $t > 0$, the solution is given by
\[
u(x, t) = \begin{cases} 
1 & x > 2t \\
\frac{1}{2} & x = 2t \\
0 & x < 2t
\end{cases}.
\]

From here, it is clear that there is a singularity at the line $x = 2t$. This is because the odd extension of $g$ is not continuous at $x = 0$. 