1 Separation of Variables I

Our goal is to develop a technique to solve IBVPs on the finite interval. This method is called *separation of variables* and it reduces the IBVP into a system of ODEs. This approach will give us a series representation of the solution, that will converge to the true solution of the IBVP.

We first explain how one would do separation of variables in the easiest case when we have a **homogeneous PDE** with **homogeneous boundary conditions**.

1. Assume the solution $u(x,t)$ is of the form $u(x,t) = X(x)T(t)$.
2. Use linearity to write the general solution as an infinite linear combination of general solutions that satisfy the PDE and boundary conditions:
   (a) Spatial Problem: Solve the spatial eigenvalue problem.
   (b) Time Problem: Solve the system of homogeneous time ODEs using the eigenvalues from the spatial problem.
3. Use the initial conditions to solve for the particular solution from the general solution using the appropriate Fourier coefficients.

1.1 Example Problems

Problem 1.1. ($\star$) Solve

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < \pi,\ t > 0 \\ u|_{t=0} = 0 & 0 < x < \pi \\ u_t|_{t=0} = x & 0 < x < \pi \\ u_x|_{x=0} = u_x|_{x=\pi} = 0 & t > 0. \end{cases}$$

Solution 1.1. This is a homogeneous PDE with homogeneous Neumann boundary conditions.

Step 1 — Separation of Variables: We look for a separated solution $u(x,t) = X(x)T(t)$ to our IBVP. Plugging this into the BVP implies

$$T''(t)X(x) - c^2 T(t)X''(x) = 0 \implies \frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

This gives the following ODEs (see Remark 1 and Remark 2)

$$X''(x) + \lambda X(x) = 0 \text{ and } T''(t) + c^2 \lambda T(t) = 0,$$

with boundary conditions

$$T(t)X'(0) = T(t)X'(\pi) = 0 \implies X'(0) = X'(\pi) = 0$$

since we can assume $T(t) \not\equiv 0$ otherwise we will have a trivial solution.

Step 2 — Spatial Problem: We begin by solving the eigenvalue problem

$$\begin{cases} -X'' = \lambda X & 0 < x < \pi \\ X'(0) = X'(\pi) = 0. \end{cases}$$

The solution to the eigenvalue problem (Week 7 Lecture Summary 1.1.2) is

Eigenvalues: $\lambda_n = n^2$ for $n = 0, 1, 2, \dots$

Eigenfunctions: $X_n = \cos(nx)$ and $X_0 = 1$.

Step 3 — Time Problem: When $n = 0$, the time problem is

$$T_0''(t) = 0$$

which has solution

$$T_0(t) = A_0 + B_0 t.$$

The time problem related to the eigenvalues λ_n for $n \geq 1$ is

$$T_n''(t) + c^2 n^2 T_n(t) = 0 \text{ for } n = 1, 2, \dots$$

which has solution

$$T_n(t) = A_n \cos(cnt) + B_n \sin(cnt).$$

Step 4 — General Solution: By the principle of superposition, the general form of our solution is

$$u(x,t) = \sum_{n=0}^{\infty} T_n(t) X_n(x) = A_0 + B_0 t + \sum_{n=1}^{\infty} \Big(A_n \cos(cnt) + B_n \sin(cnt) \Big) \cos(nx).$$

Step 5 — Particular Solution: We now use the initial conditions to recover the particular solution by solving for the constants A_n and B_n. The initial conditions imply

$$u(x,0) = 0 \implies A_0 + \sum_{n=1}^{\infty} A_n \cos(nx) = 0$$

and

$$u_t(x,0) = x \implies B_0 + \sum_{n=1}^{\infty} B_n cn \cos(nx) = x.$$

Clearly the first initial condition implies that $A_n = 0$ for all $n \geq 0$. To find the B_n coefficients, we decompose x into its Fourier cosine series (or equivalently, decomposing $|x|$ into its full Fourier series on $[-\pi, \pi]$)

$$x = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2((-1)^n - 1)}{\pi n^2} \cos(nx)$$

and equate coefficients to conclude

$$B_0 = \frac{\pi}{2}, \qquad B_n cn = \frac{2((-1)^n - 1)}{\pi n^2} \implies B_n = \frac{2((-1)^n - 1)}{c\pi n^3}.$$

Therefore, our particular solution is

$$u(x,t) = B_0 t + \sum_{n=1}^{\infty} B_n \sin(cnt) \cos(nx)$$

where $B_0 = \frac{\pi}{2}$ and $B_n = \frac{2((-1)^n - 1)}{c\pi n^3}$.

Remark 1. In general, if

$$f(x) = g(t)$$

for all x and t, then there must exist a C such that $f(x) = g(t) = C$. To see why, we can take the partial derivatives to conclude that

$$\partial_x f(x) = \partial_x g(t) = 0 \qquad \text{and} \qquad \partial_t g(t) = \partial_t f(x) = 0,$$

so both f and g must be constant by the mean value theorem. Since $f(x) = g(t)$, this implies the constant must be the same.

Remark 2. We will explain how one can rigorously justify the fact that

$$T''(t)X(x) - c^2T(t)X''(x) = 0 \implies X''(x) + \lambda X(x) = 0 \quad \text{and} \quad T''(t) + c^2\lambda T(t) = 0$$

for some λ. Since we are interested in non-trivial solutions, there exists a x_0 such that $X(x_0) \neq 0$. This implies that

$$T''(t)X(x_0) - c^2T(t)X''(x_0) = 0 \implies T''(t) - c^2\frac{X''(x_0)}{X(x_0)}T(t) = 0 \implies T''(t) + c^2\lambda T(t) = 0$$

where $\lambda_1 := -\frac{X''(x_0)}{X(x_0)}$. Similarly, for a non-trivial solution to exist, there must be a t_0 such that $T(t_0) \neq 0$, which will imply

$$T''(t_0)X(x) - c^2T(t_0)X''(x) = 0 \implies X''(x) - \frac{T''(t_0)}{c^2T(t_0)}X(x) = 0 \implies X''(x) + \lambda_2 X(x) = 0$$

where $\lambda_2 := -\frac{T''(t_0)}{c^2T(t_0)}$. To see why $\lambda_1 = \lambda_2$, we can evaluate our original equation at (x_0, t_0) and use the fact $-\lambda_1 X(x_0) = X''(x_0)$ and $-c^2\lambda_2 T(t_0) = T''(t_0)$ to see that

$$T''(t_0)X(x_0) - c^2T(t_0)X''(x_0) = 0 \implies c^2T(t_0)X(x_0)(\lambda_1 - \lambda_2) = 0 \implies \lambda_1 = \lambda_2$$

since $c^2T(t_0)X(x_0) \neq 0$. A similar procedure can be done to rigorously justify "dividing by zero" in all separation of variables problems. It also provides another proof of the appearance of the the constant in Remark 1.

Problem 1.2. ($\star\star$) Solve

$$\begin{cases} u_{tt} - c^2u_{xx} = 0 & 0 < x < L,\ t > 0 \\ u|_{t=0} = g(x) & 0 < x < L \\ u_t|_{t=0} = h(x) & 0 < x < L \\ u_x|_{x=0} = u|_{x=L} = 0 & t > 0. \end{cases}$$

Solution 1.2. This is a homogeneous problem with homogeneous boundary conditions.

Step 1 — Separation of Variables: We look for a separated solution $u(x,t) = X(x)T(t)$ to our IBVP. Plugging this into the BVP implies

$$T''(t)X(x) - kT(t)X''(x) = 0 \implies \frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

This gives the following ODEs

$$X''(x) + \lambda X(x) = 0 \text{ and } T''(t) + c^2\lambda T(t) = 0,$$

with boundary conditions

$$T(t)X'(0) = 0 = T(t)X(L) \implies X'(0) = X(L) = 0$$

since we can assume $T(t) \not\equiv 0$ otherwise we will have a trivial solution.

Step 2 — Spatial Problem: We begin by solving the eigenvalue problem

$$\begin{cases} -X'' = \lambda X & 0 < x < L \\ X'(0) = X(L) = 0. \end{cases}$$

The solution to the eigenvalue problem (Week 7 Lecture Summary 1.1.5) is

Eigenvalues:

$$\lambda_n = \Big(\frac{(2n-1)\pi}{2L}\Big)^2 \text{ for } n = 1,2,3,\dots.$$

Eigenfunctions:

$$X_n(x) = \cos\Big(\frac{(2n-1)\pi}{2L}x\Big).$$

Step 3 — Time Problem: The time problem related to the eigenvalues λ_n is

$$T_n''(t) + c^2\Big(\frac{(2n-1)\pi}{2L}\Big)^2 T_n(t) = 0 \text{ for } n = 1,2,\dots$$

which has solution

$$T_n(t) = A_n \cos\Big(\frac{c(2n-1)\pi}{2L}t\Big) + B_n \sin\Big(\frac{c(2n-1)\pi}{2L}t\Big).$$

Step 4 — General Solution: By the principle of superposition, the general form of our solution is

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)X_n(x) = \sum_{n=1}^{\infty}\bigg(A_n \cos\Big(\frac{c(2n-1)\pi}{2L}t\Big) + B_n \sin\Big(\frac{c(2n-1)\pi}{2L}t\Big)\bigg)\cos\Big(\frac{(2n-1)\pi}{2L}x\Big).$$

Step 5 — Particular Solution: We now use the initial conditions to recover the particular solution by solving for the constants A_n and B_n. The initial conditions imply

$$u(x,0) = g(x) \implies \sum_{n=1}^{\infty} A_n \cos\Big(\frac{(2n-1)\pi}{2L}x\Big) = g(x) \tag{1}$$

and

$$u_t(x,0) = h(x) \implies \sum_{n=1}^{\infty} B_n \frac{c(2n-1)\pi}{2L}\cos\Big(\frac{(2n-1)\pi}{2L}x\Big) = h(x).$$

The eigenfunction corresponding to symmetric boundary conditions are orthogonal so the coefficients are given by

$$A_n = \frac{\langle g(x), X_n(x)\rangle}{\langle X_n(x), X_n(x)\rangle} = \frac{\int_0^L g(x)\cos\Big(\frac{(2n-1)\pi}{2L}x\Big)\,dx}{\int_0^L \cos^2\Big(\frac{(2n-1)\pi}{2L}x\Big)\,dx} = \frac{2}{L}\cdot\int_0^L g(x)\cos\Big(\frac{(2n-1)\pi}{2L}x\Big)\,dx$$

and

$$\begin{aligned} B_n = \Big(\frac{c(2n-1)\pi}{2L}\Big)^{-1}\frac{\langle h(x), X_n(x)\rangle}{\langle X_n(x), X_n(x)\rangle} &= \Big(\frac{c(2n-1)\pi}{2L}\Big)^{-1}\cdot\frac{\int_0^L h(x)\cos\Big(\frac{(2n-1)\pi}{2L}x\Big)\,dx}{\int_0^L \cos^2\Big(\frac{(2n-1)\pi}{2L}x\Big)\,dx} \\ &= \Big(\frac{c(2n-1)\pi}{2L}\Big)^{-1}\cdot\frac{2}{L}\cdot\int_0^L h(x)\cos\Big(\frac{(2n-1)\pi}{2L}x\Big)\,dx. \end{aligned}$$

Remark 3. It is easy to check that these mixed boundary conditions satisfy the symmetry condition. For example, if X_1 and X_2 satisfy the boundary conditions $X_1'(0) = 0$, $X_1(L) = 0$ and $X_2'(0) = 0$, $X_2(L) = 0$ then they satisfy the symmetric condition

$$X_1'(x)X_2(x) - X_1(x)X_2'(x)\Big|_0^L = X_1'(L)X_2(L) - X_1(L)X_2'(L) - X_1'(0)X_2(0) + X_1(0)X_2'(0) = 0,$$

so the eigenfunctions of distinct eigenvalues are orthogonal and form a basis in $L^2([0,L])$.

Problem 1.3. ($\star\star$) Solve

$$\begin{cases} u_t - ku_{xx} = 0 & 0 < x < 1 \quad t > 0 \\ u|_{t=0} = g(x) & 0 < x < 1 \\ u_x|_{x=0} = (u_x + u)|_{x=1} = 0 & t > 0 \end{cases}.$$

Solution 1.3. This is a homogeneous problem with homogeneous boundary conditions.

Step 1 — Separation of Variables: We look for a separated solution $u(x,t) = X(x)T(t)$ to our PDE. Plugging this into our PDE gives

$$T'(t)X(x) - kT(t)X''(x) = 0 \implies \frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

This gives the following ODEs

$$X''(x) + \lambda X(x) = 0 \text{ and } T'(t) + k\lambda T(t) = 0,$$

with boundary conditions

$$T(t)X'(0) = 0 \text{ and } T(t)X'(1) + T(t)X(1) = 0 \implies X'(0) = X'(1) + X(1) = 0$$

since we can assume $T(t) \not\equiv 0$ otherwise we will have a trivial solution.

Step 2 — Spatial Problem: We begin by solving the eigenvalue problem

$$\begin{cases} -X'' = \lambda X & 0 < x < 1 \\ X'(0) = X'(1) + X(1) = 0. \end{cases}$$

We consider the 3 cases corresponding to the different forms of the ODE:

1. $\lambda = \beta^2 > 0$: The solution is of the form

   $$X(x) = A\cos(\beta x) + B\sin(\beta x).$$

   From the boundary conditions we get

   $$\begin{aligned} \beta B &= 0 \\ -\beta A\sin(\beta) + \beta B\cos(\beta) + A\cos(\beta) + B\sin(\beta) &= 0. \end{aligned}$$

   We can write this system of equations in matrix form

   $$\begin{bmatrix} 0 & \beta \\ \cos(\beta) - \beta\sin(\beta) & \beta\cos(\beta) + \sin(\beta) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

   which has a non-trivial solution when

   $$\begin{vmatrix} 0 & \beta \\ \cos(\beta) - \beta\sin(\beta) & \beta\cos(\beta) + \sin(\beta) \end{vmatrix} = 0 \implies \beta\cos(\beta) - \beta^2\sin(\beta) = 0.$$

   If β_n is chosen such that $\cos(\beta_n) = 0$, then $\beta_n \neq 0$ and $\sin(\beta_n) \neq 0$ which means there are no solutions such that $\cos(\beta_n) = 0$. Therefore, we can rearrange terms to recover the condition

   $$\beta\cos(\beta) - \beta^2\sin(\beta) \implies \tan(\beta) = \frac{1}{\beta}.$$

   The eigenvalues β_n are the positive roots of $\tan(\beta) = \frac{1}{\beta}$ for which there are infinitely many of them. The first boundary condition also implies $B = 0$, which means the corresponding eigenfunction of the eigenvalue $\lambda_n = \beta_n^2$ is $X_n = \cos\big(\beta_n x\big)$.

2. $\lambda = 0$: The solution is of the form
$$X(x) = A + Bx.$$
From the boundary conditions we get
$$\begin{aligned} B &= 0 \\ A + 2B &= 0. \end{aligned}$$
We can write this system of equations in matrix form
$$\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
which has has no non-trivial solutions because the first matrix is invertible. Therefore, there are no 0 eigenvalues.

3. $\lambda = -\beta^2 < 0$: The solution is of the form
$$X(x) = A\cosh(\beta x) + B\sinh(\beta x).$$
From the boundary conditions we get
$$\begin{aligned} \beta B &= 0 \\ \beta A\sinh(\beta) + \beta B\cosh(\beta) + A\cosh(\beta) + B\sinh(\beta) &= 0. \end{aligned}$$
We can write this system of equations in matrix form
$$\begin{bmatrix} 0 & \beta \\ \cosh(\beta) + \beta\sinh(\beta) & \beta\cosh(\beta) + \sinh(\beta) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
which has a non-trivial solution when
$$\begin{vmatrix} 0 & \beta \\ \cosh(\beta) + \beta\sinh(\beta) & \beta\cosh(\beta) + \sinh(\beta) \end{vmatrix} = 0 \implies \beta\cosh(\beta) + \beta^2\sinh(\beta) = 0.$$
Since $\cosh(\beta) > 0$ and $\beta > 0$, we can write the above as
$$\tanh(\beta) = -\frac{1}{\beta}$$
which has no positive roots. Therefore, there are no negative eigenvalues.

Therefore, the solution to the eigenvalue problem is

Eigenvalues: $\lambda_n = \beta_n^2$ for $n = 1, 2, \dots$ where β_n are the ordered positive roots of $\tan(\beta) = \frac{1}{\beta}$

Eigenfunctions: $X_n = \cos\big(\beta_n x\big)$.

Step 3 — Time Problem: The time problem related to the eigenvalues λ_n is
$$T_n'(t) + k(\beta_n)^2 T_n(t) = 0 \text{ for } n = 1, 2, \dots$$
which has solution
$$T_n(t) = A_n e^{-k\beta_n^2 t}.$$
Step 4 — General Solution: By the principle of superposition, the general form of our solution is
$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)X_n(x) = \sum_{n=1}^{\infty} A_n e^{-k\beta_n^2 t}\cos(\beta_n x).$$

Step 5 — Particular Solution: We now use the initial conditions to recover the particular solution by solving for the constants A_n. The initial conditions imply

$$u(x,0) = g(x) \implies \sum_{n=1}^{\infty} A_n \cos(\beta_n x) = g(x).$$

The eigenfunction corresponding to Robin boundary conditions are also symmetric boundary conditions, so the eigenfunctions are orthogonal. Therefore, the coefficients are given by

$$A_n = \frac{\langle g(x), X_n(x)\rangle}{\langle X_n(x), X_n(x)\rangle} = \frac{\int_0^1 g(x)\cos(\beta_n x)\,dx}{\int_0^1 \cos^2(\beta_n x)\,dx}.$$

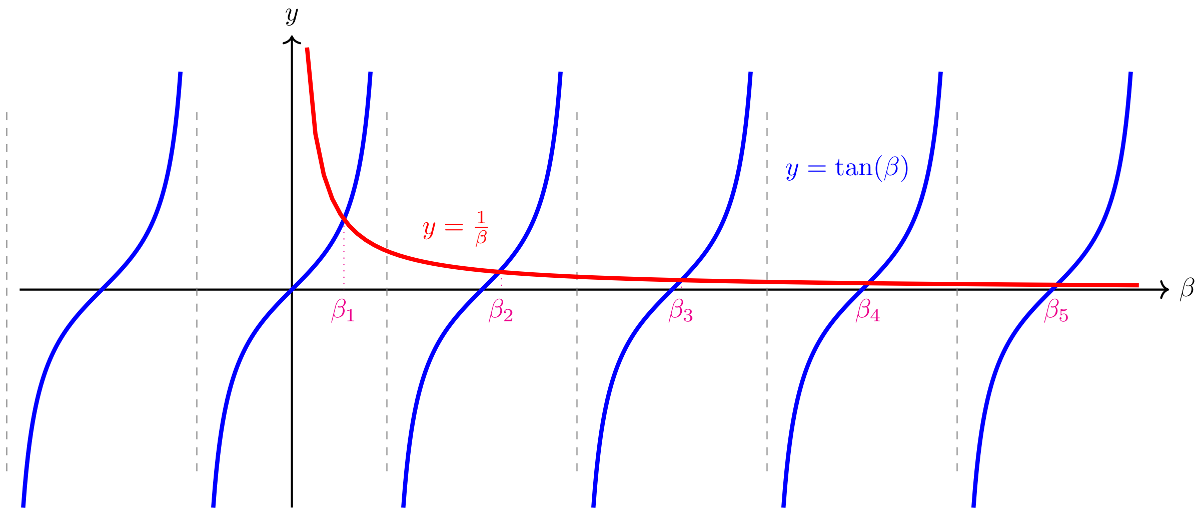

Figure 1. The values of β_n for $n = 1, \dots, 5$. Numerically, $\beta_1 \approx 0.86$, $\beta_2 \approx 3.462$, $\beta_3 \approx 6.437$, etc.

2 Separation of Variables II

We now explain how one would do separation of variables in the slightly harder case when we have a **inhomogeneous PDE** with **homogeneous boundary conditions**. We deal with the inhomogeneous PDE using the method of *eigenfunction expansions*

1. Assume the solution $u(x,t)$ is of the form $u(x,t) = X(x)T(t)$.
2. Use linearity to write the general solution of the homogeneous PDE as an infinite linear combination of general solutions that satisfy the PDE and boundary conditions:
   (a) Spatial Problem: Solve the spatial eigenvalue problem.
   (b) Time Problem: Decompose the inhomogeneous term into the Fourier series corresponding to the eigenfunctions from the spatial problem. Then solve the infinite system of inhomogeneous time ODEs using the eigenvalues from the spatial problem.
3. Use the initial conditions to solve for the particular solution from the general solution using the appropriate Fourier coefficients.

2.1 Example Problems

Problem 2.1. ($\star$) Solve

$$\begin{cases} u_t - u_{xx} + u = 1 & -\pi < x < \pi \quad t > 0 \\ u|_{t=0} = g(x) & -\pi < x < \pi \\ u|_{x=-\pi} = u|_{x=\pi} & t > 0 \\ u_x|_{x=-\pi} = u_x|_{x=\pi} & t > 0 \end{cases}$$

Solution 2.1. This is an inhomogeneous problem with homogeneous boundary conditions.

Step 1 — Separation of Variables: We first find a solution to the homogeneous equation.

$$T'(t)X(x) - T(t)X''(x) + X(x)T(t) = 0 \implies \frac{T'(t) + T(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

This gives the following ODEs

$$X''(x) + \lambda X(x) = 0 \text{ and } T'(t) + T(t) + \lambda T(t) = 0,$$

with boundary conditions

$$T(t)X'(-\pi) - T(t)X'(\pi) = 0, T(t)X(-\pi) - T(t)X(\pi) = 0 \implies X'(-\pi) - X'(\pi) = X(-\pi) - X(\pi) = 0$$

since we can assume $T(t) \not\equiv 0$ otherwise we will have a trivial solution.

Step 2 — Spatial Problem: We begin by solving the eigenvalue problem

$$\begin{cases} -X'' = \lambda X & 0 < x < 1 \\ X'(-\pi) - X'(\pi) = X(-\pi) - X(\pi) = 0. \end{cases}$$

The solution to the eigenvalue problem (Week 7 Lecture Summary 1.1.3) is

Eigenvalues: $\lambda_n = n^2$ for $n = 0, 1, 2, \ldots$

Eigenfunctions: $X_n = \cos\big(nx\big)$ and $Y_n = \sin\big(nx\big)$ and $X_0 = 1$.

Step 3 — General Homogeneous Solution: By the principle of superposition, our general solution is of the form

$$u(x,t) = T_0(t)X_0(x) + \sum_{n=1}^{\infty}\Big(T_n(t)X_n(x) + S_n(t)Y_n(x)\Big) = T_0(t) + \sum_{n=1}^{\infty}\Big(T_n(t)\cos(nx) + S_n(t)\sin(nx)\Big).$$

Step 4 — General Inhomogeneous Solution: Our goal is to now solve for $T_n(t)$ and $S_n(t)$ using the fact

$$u_t - u_{xx} + u = 1.$$

Differentiating our general solution (term by term differentiation is valid because u satisfies the periodic boundary conditions), we have

$$(T_0' + T_0) + \sum_{n=1}^{\infty}\Big((T_n' + n^2T_n + T_n)\cos(nx) + (S_n' + n^2S_n + S_n)\sin(nx)\Big) = 1.$$

To compute T_n and S_n notice that these functions play the role of the Fourier coefficients. That is, $a_0(t) = (T_0' + T_0)$, $a_n(t) = (T_n' + n^2T_n + T_n)$ and $b_n(t) = (S_n' + n^2S_n + S_n)$. If we compute the Fourier series of 1 we get only the constant term remains that is $a_0 = 1$, $a_n = 0$ and $b_n = 0$. Equating coefficients implies

$$T_0' + T_0 = 1, \quad (T_n' + n^2T_n + T_n) = 0, \quad (S_n' + n^2S_n + S_n) = 0.$$

Notice that the first ODE is a linear first order equation, so its solution is

$$T_0 = 1 + A_0e^{-t}$$

and the other ODES are separable with solutions

$$T_n = A_ne^{-(n^2+1)t}, \quad S_n = B_ne^{-(n^2+1)t}.$$

Putting this all together, we have our solution general solution to the inhomogeneous PDE is given by

$$u(x,t) = 1 + A_0e^{-t} + \sum_{n=1}^{\infty}\Big(A_ne^{-(n^2+1)t}\cos(nx) + B_ne^{-(n^2+1)t}\sin(nx)\Big).$$

Step 5 — Particular Solution: We now use the initial conditions to recover the particular solution. The initial conditions imply

$$u(x,0) = g(x) \implies 1 + A_0 + \sum_{n=1}^{\infty}\Big(A_n\cos(nx) + B_n\sin(nx)\Big) = g(x).$$

Writing $g(x)$ in terms of its full Fourier series and equating coefficients implies

$$A_0 = -1 + \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\,dx,$$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos(nx)\,dx,$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin(nx)\,dx.$$

3 Separation of Variables III

We now explain how one would do separation of variables in the hardest case when we have an **inhomogeneous PDE** with **inhomogeneous boundary conditions**. We deal with the inhomogenous boundary conditions by subtracting the inhomogenous terms away via a change of variables.

1. We define
$$u(x,t) = v(x,t) + w(x,t)$$
where $w(x,t)$ is chosen to satisfy the inhomogeneous boundary conditions. For second order PDEs, we can choose $w(x,t)$ to be a polynomial in x of the form
$$w(x,t) = (Ax^2 + Bx + C)p(t) + (Dx^2 + Ex + F)q(t),$$
for some constants $A, B, \dots F$ where $p(t)$ and $q(t)$ are the inhomogeneous boundary conditions. We choose the constants $A, B, \dots, F$ so that $w(x,t)$ solves the boundary conditions, which will imply that $v(x,t)$ solves a **inhomogeneous PDE** with **homogeneous boundary conditions**, so we can proceed as before (Separation of Variables II).
2. Assume the solution $v(x,t)$ is of the form $v(x,t) = X(x)T(t)$.
3. Use linearity to write the general solution as an infinite linear combination of general solutions that satisfy the PDE and boundary conditions:
   (a) Spatial Problem: Solve the spatial eigenvalue problem.
   (b) Time Problem: Decompose the inhomogeneous term into the Fourier series corresponding to the eigenfunctions from the spatial problem. Then solve the infinite system of inhomogeneous time ODEs using the eigenvalues from the spatial problem.
4. Use the initial conditions to solve for the particular solution from the general solution using the appropriate Fourier coefficients.

Remark 4. The approach in this section also applies to the cases in Section 1 and Section 2. In particular, it works with **homogeneous PDEs** with **inhomogeneous boundary conditions**.

3.1 Example Problems

Problem 3.1. ($\star\star$) Solve
$$\begin{cases} u_{tt} - 9u_{xx} = 0 & 0 < x < 1 \quad t > 0 \\ u|_{t=0} = g(x) & 0 < x < 1 \\ u_t|_{t=0} = h(x) & 0 < x < 1 \\ u|_{x=0} = 0 & t > 0 \\ u_x|_{x=1} = \frac{1}{2} & t > 0 \end{cases}$$

Solution 3.1. This is an homogeneous PDE with inhomogeneous mixed boundary conditions.

Step 1 — Change of Variables: Before doing separation of variables, we begin by using a change of variables to reduce our problem to the case with homogeneous boundary conditions. We set
$$u(x,t) = v(x,t) + w(x,t)$$
where $w(x,t)$ is chosen to satisfy the inhomogeneous boundary conditions. For second order PDEs, we can choose $w(x)$ to be a polynomial in x of the form
$$w(x,t) = Ax^2 + Bx + C,$$

for some constants A, B, C. Substituting $w(x)$ into the boundary conditions gives

$$C = 0 = w(0)$$
$$2A + B = \frac{1}{2} = w(1).$$

By inspection it is clear that $B = \frac{1}{2}$, $A = 0$, and $C = 0$ works. Therefore,

$$w(x) = \frac{1}{2}x.$$

Step 2 — Separation of Variables: For this choice of $w(x)$, the function $v(x,t) = u(x,t) - w(x)$ satisfies the following homogeneous PDE with homogeneous boundary conditions

$$\begin{cases} v_{tt} - 9v_{xx} = 0 & 0 < x < 1 \quad t > 0 \\ v|_{t=0} = g(x) - \frac{1}{2}x & 0 < x < 1 \\ v_t|_{t=0} = h(x) & 0 < x < 1 \\ v|_{x=0} = 0, \quad v_x|_{x=1} = 0 & t > 0 \end{cases}$$

We look for a solution of the form $v(x,t) = X(x)T(t)$. For such a solution, the PDE implies

$$T'X - kTX'' = 0 \implies \frac{T''}{3^2 T} = \frac{X''}{X} = -\lambda.$$

This results in the ODEs

$$X''(x) + \lambda X(x) = 0, \quad T''(t) + 3^2\lambda T(t) = 0$$

with boundary conditions

$$T(t)X(0) = 0 = T(t)X'(1).$$

For non-trivial solutions, we can require $T(t) \not\equiv 0$, $X(0) = X'(1) = 0$.

Step 3 — Eigenvalue Problem: We solve the spatial eigenvalue problem

$$\begin{cases} -X'' = \lambda X & 0 < x < \pi \\ X(0) = X'(1) = 0. \end{cases}$$

The eigenvalues and corresponding eigenfunctions (Week 7 Lecture Summary 1.1.4) are

$$\lambda_n = \Big(\frac{(2n-1)\pi}{2}\Big)^2, \ X_n(x) = \sin\Big(\frac{(2n-1)\pi}{2}x\Big), \quad n = 1, 2, 3, \ldots$$

Step 4 — Time Problem: The time problem related to the eigenvalues λ_n is

$$T_n''(t) + 3^2\Big(\frac{(2n-1)\pi}{2}\Big)^2 T_n(t) = 0 \text{ for } n = 1, 2, \ldots$$

which has solution

$$T_n(t) = A_n \cos\Big(\frac{3(2n-1)\pi}{2}t\Big) + B_n \sin\Big(\frac{3(2n-1)\pi}{2}t\Big).$$

where A_n and B_n are yet to be determined constants. Taking the linear combination of T_n with the eigenfunctions imply our general solution is of the form,

$$v(x,t) = \sum_{n=1}^{\infty} \bigg(A_n \cos\Big(\frac{3(2n-1)\pi}{2}t\Big) + B_n \sin\Big(\frac{3(2n-1)\pi}{2}t\Big)\bigg) \sin\Big(\frac{(2n-1)\pi}{2}x\Big)$$

Step 5 — Particular Solution: The initial conditions imply

$$v(x,0) = g(x) - \frac{x}{2} \implies \sum_{n=1}^{\infty} A_n \sin\left(\frac{(2n-1)\pi}{2}x\right) = g(x) - \frac{x}{2}$$

and

$$v_t(x,0) = h(x) \implies \sum_{n=1}^{\infty} B_n \frac{3(2n-1)\pi}{2} \sin\left(\frac{(2n-1)\pi}{2L}x\right) = h(x).$$

The eigenfunction corresponding to symmetric boundary conditions are orthogonal so the coefficients are given by

$$\begin{aligned} A_n = \frac{\langle g(x) - \frac{x}{2}, X_n(x)\rangle}{\langle X_n(x), X_n(x)\rangle} &= \frac{\int_0^1 \left(g(x) - \frac{x}{2}\right) \sin\left(\frac{(2n-1)\pi}{2}x\right) dx}{\int_0^1 \sin^2\left(\frac{(2n-1)\pi}{2}x\right) dx} \\ &= 2 \cdot \int_0^1 \left(g(x) - \frac{x}{2}\right) \sin\left(\frac{(2n-1)\pi}{2}x\right) dx \end{aligned}$$

and

$$\begin{aligned} B_n = \left(\frac{3(2n-1)\pi}{2}\right)^{-1} \frac{\langle h(x), X_n(x)\rangle}{\langle X_n(x), X_n(x)\rangle} &= \left(\frac{3(2n-1)\pi}{2}\right)^{-1} \cdot \frac{\int_0^1 h(x) \sin\left(\frac{(2n-1)\pi}{2}x\right) dx}{\int_0^1 \sin^2\left(\frac{(2n-1)\pi}{2}x\right) dx} \\ &= \frac{4}{3(2n-1)\pi} \cdot \int_0^1 h(x) \sin\left(\frac{(2n-1)\pi}{2}x\right) dx. \end{aligned}$$

Step 6 — Final Answer: We now summarize our solution. Recalling that $u = v + w$, we have

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{3(2n-1)\pi}{2}t\right) + B_n \sin\left(\frac{3(2n-1)\pi}{2}t\right)\right) \sin\left(\frac{(2n-1)\pi}{2}x\right) + \frac{1}{2}x$$

where the coefficients are given by

$$A_n = 2 \cdot \int_0^1 \left(g(x) - \frac{x}{2}\right) \sin\left(\frac{(2n-1)\pi}{2}x\right) dx, \quad B_n = \frac{4}{3(2n-1)\pi} \cdot \int_0^1 h(x) \sin\left(\frac{(2n-1)\pi}{2}x\right) dx.$$

Problem 3.2. ($\star\star$) Solve

$$\begin{cases} u_t - ku_{xx} = e^{-x} & 0 < x < \pi \qquad t > 0 \\ u|_{t=0} = g(x) & 0 < x < \pi \\ u_x|_{x=0} = 1 & t > 0 \\ u_x|_{x=\pi} = 0 & t > 0. \end{cases}$$

Solution 3.2. This is an inhomogeneous PDE with inhomogeneous Neumann boundary conditions.

Step 1 — Change of Variables: Before doing separation of variables, we begin by using a change of variables to reduce our problem to the case with homogeneous boundary conditions. We set

$$u(x,t) = v(x,t) + w(x,t)$$

where $w(x,t)$ is chosen to satisfy the inhomogeneous boundary conditions. For second order PDEs, we can choose $w(x)$ to be a polynomial in x of the form

$$w(x,t) = Ax^2 + Bx + C,$$

for some constants A, B, C. Substituting $w(x)$ into the boundary conditions gives

$$\begin{aligned} B = 1 &= w(0,t) \\ 2\pi A + B = 0 &= w(\pi,t). \end{aligned}$$

By inspection it is clear that $B = 1$, $C = -\frac{1}{2\pi}$, and $C = 0$ works. Therefore,

$$w(x) = -\frac{1}{2\pi}x^2 + x.$$

Step 2 — Separation of Variables: For this choice of $w(x)$, the function $v(x,t) = u(x,t) - w(x)$ satisfies the following inhomogeneous PDE with homogeneous boundary conditions

$$\begin{cases} v_t - kv_{xx} = e^{-x} - k\frac{1}{\pi} & 0 < x < \pi \quad t > 0 \\ v|_{t=0} = g(x) + \frac{1}{2\pi}x^2 - x & 0 < x < \pi \\ v_x|_{x=0} = v_x|_{x=\pi} = 0 & t > 0. \end{cases}$$

We look for a solution of the form $v(x,t) = X(x)T(t)$. For such a solution, the PDE implies

$$T'X - kTX'' = 0 \implies \frac{T'}{kT} = \frac{X''}{X} = -\lambda.$$

This results in the ODEs

$$X''(x) + \lambda X(x) = 0,$$

with boundary conditions

$$T(t)X'(0) = 0 = T(t)X'(\pi).$$

For non-trivial solutions, we can require $T(t) \not\equiv 0$, $X'(0) = X'(\pi) = 0$.

Step 3 — Eigenvalue Problem: We solve the spatial eigenvalue problem

$$\begin{cases} -X'' = \lambda X & 0 < x < \pi \\ X'(0) = X'(\pi) = 0. \end{cases}$$

This is a standard eigenvalue problem and the eigenvalues and corresponding eigenfunctions are

$$\lambda_0 = 0, \ X_0 = 1 \quad \lambda_n = n^2, \ X_n(x) = \cos(nx), \quad n = 1, 2, 3, \ldots$$

Step 4 — Time Problem: We now use the method of eigenfunction expansion to find $T_n(t)$ that satisfies the inhomogeneous equation. By the principle of superposition, the general solution to the homogeneous PDE is of the form

$$v(x,t) = T_0 + \sum_{n=1}^{\infty} T_n(t)\cos(nx).$$

Differentiating term by term (valid since the boundary conditions are homogeneous) and plugging this into our inhomogeneous PDE gives

$$v_t - kv_{xx} = T_0' + \sum_{n=1}^{\infty} T_n'(t)\cos(nx) + k\sum_{n=1}^{\infty} T_n(t)n^2\cos(nx) = e^{-x} - k\frac{1}{\pi}.$$

We write the right hand side of the above equation as the Fourier cosine series

$$e^{-x} - k\frac{1}{\pi} = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx)$$

where

$$a_n = \frac{2}{\pi}\int_0^\pi \Big(e^{-x} - k\frac{1}{\pi}\Big)\cos(nx)\,dx = \frac{2}{\pi}\cdot\frac{(-1)^{n+1}e^{-\pi}+1}{n^2+1}$$

and

$$a_0 = \frac{1}{\pi}\int_0^\pi \Big(e^{-x} - k\frac{1}{\pi}\Big)\,dx = \frac{-k+1+\sinh(\pi)-\cosh(\pi)}{\pi}.$$

Equating coefficients, we have for $n \geq 1$,

$$T_0'(t) = a_0 \quad T_n'(t) + kn^2T_n(t) = a_n.$$

This is a first order linear ODE. Its solution can be found using an integrating factor of the form e^{kn^2t}, leading to the general solution

$$T_0(t) = a_0t + A_0, \quad T_n(t) = A_ne^{-kn^2t} + \int_0^t a_ne^{-kn^2(t-s)}\,ds = A_ne^{-kn^2t} + \frac{a_n}{kn^2} - \frac{a_ne^{-kn^2t}}{kn^2}.$$

where A_0, A_n are yet to be determined constants.

Step 5 — Particular Solution: We now use the initial conditions to determine A_0, and A_n. The initial conditions imply

$$v(x,0) = A_0 + \sum_{n=1}^\infty A_n\cos(nx) = g(x) + \frac{1}{2\pi}x^2 - x.$$

The coefficients A_0, A_n are the coefficients of the Fourier cosine series of $g(x) + \frac{1}{2\pi}x^2 - x$, which is given explicitly by

$$A_0 = \frac{1}{\pi}\int_0^\pi \Big(g(x) + \frac{1}{2\pi}x^2 - x\Big)\,dx \qquad A_n = \frac{2}{\pi}\int_0^\pi \Big(g(x) + \frac{1}{2\pi}x^2 - x\Big)\cos(nx)\,dx.$$

Step 6 — Final Answer: We now summarize our solution. Recalling that $u = v + w$, we have

$$\begin{aligned} u(x,t) &= \frac{-k+1+\sinh(\pi)-\cosh(\pi)}{\pi}\cdot t + A_0 \\ &\quad + \sum_{n=1}^\infty \Big(A_n\cdot e^{-kn^2t} + \frac{2}{\pi kn^2}\cdot\frac{(-1)^{n+1}e^{-\pi}+1}{n^2+1}(1-e^{-kn^2t})\Big)\cos(nx) - \frac{1}{2\pi}x^2 + x \end{aligned}$$

where A_0 and A_n are given by

$$A_0 = \frac{1}{\pi}\int_0^\pi \Big(g(x) + \frac{1}{2\pi}x^2 - x\Big)\,dx \quad A_n = \frac{2}{\pi}\int_0^\pi \Big(g(x) + \frac{1}{2\pi}x^2 - x\Big)\cos(nx)\,dx.$$

Problem 3.3. ($\star\star\star$) Solve

$$\begin{cases} u_t - ku_{xx} = 0 & 0 < x < 1 \quad t > 0 \\ u|_{t=0} = x & 0 < x < 1 \\ u|_{x=0} = \sin(t) & t > 0 \\ (u_x + u)|_{x=1} = 2 & t > 0. \end{cases}$$

Solution 3.3. This is an homogeneous PDE with time dependent boundary conditions.

Step 1 — Change of Variables: Before doing separation of variables, we begin by using a change of variables to reduce our problem to the case with homogeneous boundary conditions. We set

$$u(x,t) = v(x,t) + w(x,t)$$

where $w(x,t)$ is chosen to satisfy the inhomogeneous boundary conditions. For second order PDEs, we can choose $w(x,t)$ to be a polynomial in x of the form

$$w(x,t) = (Ax^2 + Bx + C)\sin(t) + (Dx^2 + Ex + F)2,$$

for some constants $A, B, \dots F$. Substituting $w(x,t)$ into the boundary conditions gives

$$\begin{aligned} C\sin(t) + 2F &= \sin(t) &&= w(0,t) \\ (3A + 2B + C)\sin(t) + (3D + 2E + F)2 &= 2 &&= w_x(1,t) + w(1,t). \end{aligned}$$

By inspection it is clear that $C = 1$, $B = \frac{-1}{2}$, and $E = \frac{1}{2}$ with the rest of the coefficients zero works. Therefore,

$$w(x,t) = (-2^{-1}x + 1)\sin(t) + (2^{-1}x)2 = \frac{2 - \sin(t)}{2}x + \sin(t).$$

Step 2 — Separation of Variables: Since $v(x,t) = u(x,t) - w(x,t)$, our choice of $w(x,t)$ implies

$$\begin{cases} v_t - kv_{xx} = \frac{\cos(t)}{2}x - \cos(t) & 0 < x < 1 \quad t > 0 \\ v|_{t=0} = 0 & 0 < x < 1 \\ v|_{x=0} = (v_x + v)|_{x=1} = 0 & t > 0. \end{cases}$$

This is an inhomogeneous PDE with homogeneous boundary conditions. We begin by using separation of variables to solve the homoegenous PDE. We look for a solution of the form $v(x,t) = T(t)X(x)$. For such a solution, the PDE implies

$$T'X - kTX'' = 0 \implies \frac{T'}{kT} = \frac{X''}{X} = -\lambda.$$

This results in the ODE

$$X''(x) + \lambda X(x) = 0,$$

with boundary conditions

$$T(t)X(0) = T(t)X'(1) + T(t)X(1) = 0$$

For non-trivial solutions, we can require $T(t) \not\equiv 0$, $X(0) = X'(1) + X(1) = 0$.

Step 3 — Eigenvalue Problem: We solve the spatial eigenvalue problem

$$\begin{cases} -X'' = \lambda X & 0 < x < 1 \\ X(0) = X'(1) + X(1) = 0. \end{cases}$$

This eigenvalue problem can be solved similarly to the one in Problem 1.3. The eigenvalues and corresponding eigenfunctions are given by

$$\lambda_n = \beta_n^2, \qquad X_n(x) = \sin(\beta_n x), \quad n = 1, 2, 3, \dots$$

where β_n are the ordered positive roots of

$$\tan(\beta) = -\beta.$$

Since the boundary conditions are symmetric, we have that the eigenfunctions $\sin(\beta_n x)$ are orthogonal.

Step 4 — Time Problem: We now use the method of eigenfunction expansion to find $T_n(t)$ that satisfies the inhomogeneous equation. By the principle of superposition, the general solution to the homogeneous PDE is of the form

$$v(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(\beta_n x).$$

Differentiating term by term (valid since the boundary conditions are homogeneous) and plugging this into our inhomogeneous PDE gives

$$v_t - kv_{xx} = \sum_{n=1}^{\infty} T_n'(t)\sin(\beta_n x) + k\sum_{n=1}^{\infty} T_n(t)\beta_n^2\sin(\beta_n x) = -\frac{\cos(t)}{2}x + \cos(t).$$

We fix t and write the right hand side of the above equation as the generalized Fourier series

$$-\frac{\cos(t)}{2}x + \cos(t) = \sum_{n=1}^{\infty} b_n(t)\sin(\beta_n x)$$

where

$$b_n(t) = \frac{\int_0^1 \left(\frac{\cos(t)}{2}x - \cos(t)\right)\sin(\beta_n x)\,dx}{\int_0^1 \sin^2(\beta_n x)\,dx}.$$

Equating coefficients, we have for $n \geq 1$,

$$T_n'(t) + k\beta_n^2 T_n(t) = b_n(t).$$

This is a first order linear ODE. Its solution can be found using an integrating factor of the form e^{kn^2t}, leading to the general solution

$$T_n(t) = C_n e^{-k\beta_n^2 t} + \int_0^t b_n(s)\exp(-k\beta_n^2(t-s))\,ds.$$

where C_n is a yet to be determined constant.

Step 5 — Particular Solution: We now use the initial conditions to determine C_n. The initial conditions imply

$$v(x,0) = \sum_{n=1}^{\infty} C_n\sin(\beta_n x) = 0.$$

Clearly we must have $C_n = 0$ for all n.

Step 6 — Final Answer: We now summarize our solution. Recalling that $u = v + w$, we have

$$u(x,t) = \sum_{n=1}^{\infty}\left(\int_0^t b_n(s)\exp(-k\beta_n^2(t-s))\,ds\right)\sin(\beta_n x) + \frac{2-\sin(t)}{2}x + \sin(t),$$

where β_n are the ordered positive roots of $\tan(\beta) = -\beta$ and

$$b_n(s) = \frac{\int_0^1 \left(\frac{\cos(s)}{2}x - \cos(s)\right)\sin(\beta_n x)\,dx}{\int_0^1 \sin^2(\beta_n x)\,dx}.$$