1 Fourier Series

We begin by introducing the *Fourier series* of a function. The series representations of solutions to PDEs on a finite interval will be expressed in the form of these series. The series coefficients are determined by the initial conditions of the problems. The Fourier series of f is of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big)\Big). \tag{1}$$

There are 3 main types of coefficients:

1. Full Fourier Series: The coefficients of the (full) Fourier series of $f : [-L, L] \to \mathbb{R}$ is given by

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\Big(\frac{n\pi x}{L}\Big)\,dx \quad \text{and} \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\Big(\frac{n\pi x}{L}\Big)\,dx. \tag{2}$$

   The partial sums of this Fourier series satisfies periodic boundary conditions, $f(-L) = f(L)$ and $f'(-L) = f'(L)$.

2. Fourier Cosine Series: The coefficients of the Fourier cosine series of $f : [0, L] \to \mathbb{R}$ is given by the coefficients of the full Fourier series of the even extension of f:

$$a_n = \frac{1}{L}\int_{-L}^{L} f_{even}(x)\cos\Big(\frac{n\pi x}{L}\Big)\,dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\Big(\frac{n\pi x}{L}\Big)\,dx \tag{3}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f_{even}(x)\sin\Big(\frac{n\pi x}{L}\Big)\,dx = 0. \tag{4}$$

   To compute b_n, we used the fact that the product of an even and odd function is odd. The partial sums of this Fourier series satisfies Neumann boundary conditions $f'(0) = f'(L) = 0$.

3. Fourier Sine Series: The coefficients of the Fourier sine series of $f : [0, L] \to \mathbb{R}$ is given by the coefficients of the full Fourier series of the odd extension of f:

$$a_n = \frac{1}{L}\int_{-L}^{L} f_{odd}(x)\cos\Big(\frac{n\pi x}{L}\Big)\,dx = 0 \tag{5}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f_{odd}(x)\sin\Big(\frac{n\pi x}{L}\Big)\,dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\Big(\frac{n\pi x}{L}\Big)\,dx. \tag{6}$$

   To compute a_n, we used the fact that the product of an even and odd function is odd. The partial sums of this Fourier series satisfies Dirichlet boundary conditions $f(0) = f(L) = 0$.

1.1 Derivation of the Fourier Series Using Orthogonality

Suppose $(X_n)_{n\geq 1}$ is an orthogonal sequence of functions on $[a, b]$. The general Fourier series of f is a series of the form

$$f(x) = \sum_{n=1}^{\infty} c_n X_n(x).$$

To solve for the Fourier coefficient c_k, we can take the inner product of both sides by X_k

$$\langle f, X_k\rangle = \Big\langle \sum_{n=1}^{\infty} c_n X_n, X_k \Big\rangle = \sum_{n=1}^{\infty} c_n \langle X_n, X_k\rangle = c_k\langle X_k, X_k\rangle \implies c_k = \frac{\langle f, X_k\rangle}{\langle X_k, X_k\rangle}. \tag{7}$$

since orthogonality implies that $\langle X_n, X_k\rangle = 0$ unless $n = k$.

We can use this formula to derive the Fourier coefficients of the full Fourier series. By the product sum identities, it is easy to check that $X_n = \cos(\frac{n\pi x}{L})$ and $Y_n = \sin(\frac{n\pi x}{L})$ for $n \geq 1$ and $X_0 = \frac{1}{2}$ for $n = 0$ are orthogonal. The full Fourier series is of $f : [-L, L] \to \mathbb{R}$ is of the form

$$f(x) = a_0 X_0(x) + \sum_{n=1}^{\infty} \big(a_n X_n(x) + b_n Y_n(x)\big) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big)\Big).$$

Its coefficients for $n \geq 1$ are

$$a_n = \frac{\langle f, X_n\rangle}{\langle X_n, X_n\rangle} = \frac{\int_{-L}^{L} f(x) X_n(x)\,dx}{\int_{-L}^{L} X_n^2(x)\,dx} = \frac{\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})\,dx}{\int_{-L}^{L} \cos^2(\frac{n\pi x}{L})\,dx} = \frac{1}{L}\int_{-L}^{L} f(x)\cos\Big(\frac{n\pi x}{L}\Big)\,dx$$

$$b_n = \frac{\langle f, Y_n\rangle}{\langle Y_n, Y_n\rangle} = \frac{\int_{-L}^{L} f(x) Y_n(x)\,dx}{\int_{-L}^{L} Y_n^2(x)\,dx} = \frac{\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})\,dx}{\int_{-L}^{L} \sin^2(\frac{n\pi x}{L})\,dx} = \frac{1}{L}\int_{-L}^{L} f(x)\sin\Big(\frac{n\pi x}{L}\Big)\,dx$$

and the coefficient for a_0 is given by

$$a_n = \frac{\langle f, X_0\rangle}{\langle X_0, X_0\rangle} = \frac{\int_{-L}^{L} f(x) X_0(x)\,dx}{\int_{-L}^{L} X_0^2(x)\,dx} = \frac{\int_{-L}^{L} f(x)\frac{1}{2}\,dx}{\int_{-L}^{L} \frac{1}{2^2}\,dx} = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

which is what we get if we naively extend the formula for a_n to a_0.

Remark 1. The coefficients of the Fourier cosine and sine series can be derived in a similar way. We can also use symmetry and the uniqueness of the Fourier coefficients to recover these coefficients from the full Fourier series.

Remark 2. It turns out that the Fourier series form a orthogonal basis on the space of square integrable functions $L^2([-L, L])$. This means that every square integrable function can be written as a linear combination of its orthogonal basis elements. The Fourier coefficients are an explicit formula for the scalars in this linear combination.

1.2 Derivation of the Fourier Series Using Least-Square Approximation

Suppose $(X_n)_{n\geq 1}$ is an orthogonal sequence of functions on $[a, b]$. The general Fourier series of f is a series of the form

$$f(x) = \sum_{n=1}^{\infty} c_n X_n(x).$$

For fixed $N \geq 1$, we want to find coefficients (c_n) that minimizes the mean-squared error

$$E_N = E_N(c_1, \dots, c_N) = \Big\| f - \sum_{n=1}^{N} c_n X_n \Big\|_{L^2}^2 = \int_{-L}^{L} \Big| f(x) - \sum_{n=1}^{N} c_n X_n(x) \Big|^2\,dx.$$

We used the notation $\|\cdot\|_{L^2}^2 = \langle\cdot,\cdot\rangle$ to denote the inner product norm. If we expand the square terms, then

$$\begin{aligned} E_N &= \int_{-L}^{L} |f(x)|^2\,dx - 2\sum_{n=1}^{N} c_n \int_{-L}^{L} f(x) X_n(x)\,dx + \sum_{n,m=1}^{N} c_n c_m \int_{-L}^{L} X_n(x) X_m(x)\,dx. \\ &= \langle f, f\rangle - 2\sum_{n=1}^{N} c_n \langle f, X_n\rangle + \sum_{n=1}^{N} c_n^2 \langle X_n, X_n\rangle \end{aligned}$$

since $\langle X_n, X_m \rangle = 0$ if $n \neq m$ by orthogonality. We can now differentiate with respect to $c_1, \dots, c_N$ to find critical point conditions,

$$\partial_{c_k} E_N = -2\langle f, X_k \rangle + 2c_k \langle X_k, X_k \rangle = 0 \implies c_k = \frac{\langle f, X_k \rangle}{\langle X_k, X_k \rangle}$$

which agrees with the formula in (7). It is easy to see that we have a minimum at the critical point, because the Hessian of E_N is positive definite because it is a diagonal matrix with diagonal entries $\langle X_k, X_k \rangle = \|X_k\|_{L^2}^2 > 0$.

1.3 Example Problems

Problem 1.1. Decompose the following functions into its Fourier series on the interval $[-1, 1]$ and sketch the graph of the sum of the first three nonzero terms of its Fourier series.

(a) $f(x) = x$

(b) $f(x) = |x|$

Solution 1.1.

(Part a) We find the Fourier coefficients defined in (2):

a_n: Since $f(x) = x$ is odd, the a_n coefficients are zero.

b_n: Using integration by parts,

$$b_n = \int_{-1}^{1} x \sin(n\pi x)\, dx = 2\int_0^1 x \sin(n\pi x)\, dx = -\frac{2(-1)^n}{\pi n}.$$

The corresponding Fourier series (1) of x is given by

$$x = \sum_{n=1}^{\infty} b_n \sin(n\pi x) = -\sum_{n=1}^{\infty} \frac{2(-1)^n}{\pi n} \sin(n\pi x).$$

(Part b) We find the Fourier coefficients defined in (2):

a_0: A simple computation shows

$$a_0 = \int_{-1}^{1} |x|\, dx = 2\int_0^1 x\, dx = 1.$$

a_n: For $n \geq 1$, we can integration by parts,

$$a_n = \int_{-1}^{1} |x| \cos(n\pi x)\, dx = 2\int_0^1 x \cos(n\pi x)\, dx = \frac{2((-1)^n - 1)}{\pi^2 n^2}.$$

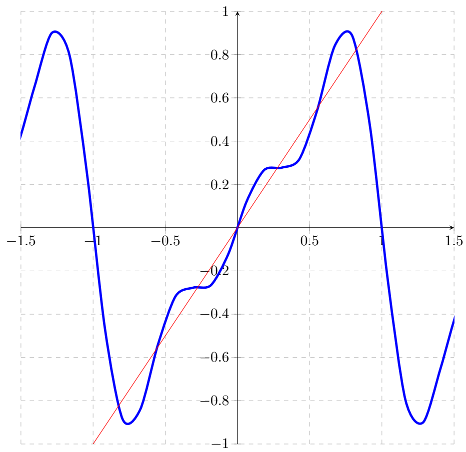
We had to treat the a_0 case separately, because we would've divided by 0 in the computation above if $n = 0$.

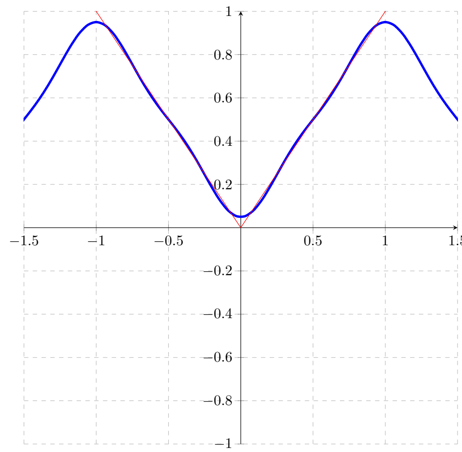
b_n: Since $f(x) = |x|$ is even, the b_n coefficients are zero.

The corresponding Fourier series (1) of $|x|$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2((-1)^n - 1)}{\pi^2 n^2} \cos(n\pi x).$$

Plots: The blue plots of the first 3 non-zero terms of the series are displayed below:

(a) 

(b) 

Remark 3. The series in part (a) and part (b) are also the respective Fourier sine and cosine series of $f(x) = x$ on $[0, 1]$ (shown in red).

Remark 4. It appears that the partial sums of the Fourier series give a good approximation of the function on $[-1, 1]$. The blue lines seem to converge to the periodic extensions of f to $\mathbb{R}$. We will show in the next section that the Fourier series do converge to the original function f almost everywhere.

The Fourier series in part (b) appears to approximate the original solution much better than the Fourier series in part (a). The Fourier series in part (b) actually converges uniformly, which is a stronger form of convergence, which states that all points are close to the original function when we take enough terms in the sum.

2 Convergence of Fourier Series

Let $(X_n)_{n\geq 1}$ be a sequence of orthogonal functions. Recall that its Fourier series is given by

$$f(x) = \sum_{n=1}^{\infty} c_n X_n(x) \qquad \text{where} \qquad c_n = \frac{\langle f, X_n \rangle}{\langle X_n, X_n \rangle}.$$

The equality here is misleading, because it might not be true that $f(x) = \sum_{n=1}^{\infty} a_n X_n(x)$ for all points x. In fact, it might not even be true that the sum on the right converges. In this section, we will go over several classes of functions and orthogonal functions such that the equality is meaningful.

2.1 Modes of Convergence

Let (f_n) be a sequence of functions on $[a, b]$. We are interested in 3 notions of convergence

1. Pointwise: We say f_n converges to f *pointwise* on an interval $I \subseteq [a, b]$ if for every $x \in I$,

$$\lim_{n\to\infty} |f_n(x) - f(x)| = 0.$$

   This says that for every point $x \in I$, $f_n(x)$ eventually gets close to $f(x)$. How fast $f_n(x)$ approaches $f(x)$ might depend on the point we pick.

2. Uniform: We say f_n converges to f *uniformly* on $[a, b]$ if

$$\lim_{n\to\infty} \max_{x\in[a,b]} |f_n(x) - f(x)| = \lim_{n\to\infty} \|f_n(x) - f(x)\|_\infty = 0.$$

   This says that the maximum distance between $f_n(x)$ and $f(x)$ get small, so f_n is close to f for all points in $[a, b]$.

3. Mean-Square: We say f_n converges to f in L^2 if

$$\lim_{n\to\infty} \int_a^b |f_n(x) - f(x)|^2 = \lim_{n\to\infty} \|f_n(x) - f(x)\|_{L^2}^2 = 0.$$

   This says that on average, the squared distance between f_n and f is small. It is possible that $f_n(x) \not\to f(x)$ at some points in this notion of convergence.

It is easy to check that uniform convergence implies pointwise convergence and L^2 convergence. The Fourier series converges if its partial sums converge, i.e.

$$f_N = \sum_{n=1}^{N} a_k X_k(x) \to f$$

in one of the 3 notations of convergence above. Since this is the "best" notion of convergence is uniform convergence, we will state a theorem that will allow us to check if an infinite series converges uniformly.

Theorem 1 (*Weierstrass M-test*)

If (f_n) is a sequence of functions on $[a, b]$ satisfying

$$\max_{x\in[a,b]} |f_n(x)| = \|f_n\|_\infty \leq M_n$$

and $\sum_{n=1}^{\infty} M_n < \infty$ then the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[a, b]$.

Proof. We prove convergence directly using the definition of uniform convergence. Let $\epsilon > 0$.

Finding a Candidate for the Limit: Since $\sum_{n=1}^{\infty} M_n < \infty$ and $M_n \geq 0$, the series converges so there exists a N_ϵ such that for all $M > N > N_\epsilon$,

$$\sum_{n=N}^{M} M_n \leq \sum_{n=N}^{\infty} M_n \leq \frac{\epsilon}{2}.$$

Therefore, for all $M > N > N_\epsilon$, the triangle inequality implies that for any $x \in [a, b]$,

$$\left| \sum_{n=1}^{M} f_n(x) - \sum_{n=1}^{N} f_n(x) \right| = \left| \sum_{n=N}^{M} f_n(x) \right| \leq \sum_{n=N}^{M} |f_n(x)| \leq \sum_{n=N}^{M} M_n \leq \frac{\epsilon}{2}.$$

We have shown that the partial sums are a Cauchy sequence, so the partial sums converge pointwise to some function $f(x)$.

Uniform Convergence: We now prove that this convergence to f is uniform. For any $x \in [a, b]$, $\sum_{n=1}^{M} f_n(x)$ converges to $f(x)$ so there exists a $M_{x,\epsilon} > N_\epsilon$ such that $|\sum_{n=1}^{M} f_n(x) - f(x)| \leq \frac{\epsilon}{2}$ for all $M > M_{x,\epsilon}$. Therefore, for $N > N_\epsilon$ and $M > \max(N, M_{x,\epsilon})$,

$$\left| f(x) - \sum_{n=1}^{N} f_n(x) \right| \leq \left| f(x) - \sum_{n=1}^{M} f_n(x) \right| + \left| \sum_{n=1}^{M} f_n(x) - \sum_{n=1}^{N} f_n(x) \right| \leq \epsilon$$

The upperbound is independent of x, so for any $N > N_\epsilon$, we have

$$\max_{x \in [a,b]} \left| f(x) - \sum_{n=1}^{N} f_n(x) \right| \leq \epsilon,$$

so the convergence is uniform. □

Uniform convergence is also a standard condition for the validity of term by term integration.

Theorem 2 (*Term by Term Integration*)

If $\sum_{n=1}^{\infty} f_n(x) = f(x)$ uniformly on $[a, b]$ and f_n are continuous on $[a, b]$ then the sum is continuous on $[a, b]$ and

$$\sum_{n=1}^{\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx.$$

Term by term differentiation is not as well behaved, so we need to assume a bit more. We essentially need to ensure that the derivative converges uniformly and that the original series can be recovered after integrating term by term.

Theorem 3 (*Term by Term Differentiation*)

If $\sum_{n=1}^{\infty} f_n(c) = f(c)$ at some $c \in [a, b]$ and $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly on $[a, b]$, then $\sum_{n=1}^{\infty} f_n(x)$ converges to some function $f(x)$ uniformly on $[a, b]$ and

$$\sum_{n=1}^{\infty} f_n'(x)\, dx = f'(x).$$

2.2 Convergence of Full Fourier Series

We now summarize some convergence results for the full Fourier series of a function $f : [-L, L] \to \mathbb{R}$. Given a function f, we denote the Nth partial sum of the Fourier series with

$$f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \right),$$

where a_0, (a_n) and (b_n) are the corresponding Fourier coefficients of f defined in (2).

2.2.1 Pointwise Convergence

Let

$$f_{ext}(x) = f\left(x - \left(\left\lceil \frac{x+L}{2L} - 1 \right\rceil (2L)\right)\right)$$

denote the $2L$ periodic extension of f. Under some mild conditions on the derivatives of f, the Fourier series converges to the function f whenever f is continuous, and to its midpoint at the jump discontinuities.

Theorem 4 (*Pointwise Convergence of Fourier Series*)

Suppose f and f' are piecewise continuous on $[-L, L]$. The full Fourier series of f converges pointwise to $\frac{f_{ext}(x^+)+f_{ext}(x^-)}{2}$. In particular, for every $x \in \mathbb{R}$

$$f_N(x) = a_0 + \sum_{n=1}^{N} \left(a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \right) \to \frac{f_{ext}(x^+) + f_{ext}(x^-)}{2}$$

Remark 5. This theorem implies that if f is continuous on $[-L, L]$ and f' is piecewise continuous on $[-L, L]$, then the full Fourier series of f converges pointwise to f on $(-L, L)$.

2.2.2 Uniform Convergence

Under some mild conditions on the derivatives of f, the Fourier series converges uniformly whenever its periodic extension is continuous.

Theorem 5 (*Uniform Convergence of Fourier Series*)

If f is continuous on $[-L, L]$ and f' is piecewise continuous on $[-L, L]$ and $f(-L) = f(L)$ then its full Fourier series converges uniformly to f on $[-L, L]$.

By the uniform limit theorem, the uniform limit of continuous functions are continuous. This implies that the continuity condition on f is necessary.

Corollary 1 (*Uniform Limit Theorem*)

If f is discontinuous on $[-L, L]$ then its Fourier series cannot converge uniformly to f on $[-L, L]$.

2.2.3 L^2 Convergence

We can show convergence in L^2 under some very weak assumptions on f. We don't need to anything about the continuity of f or its derivatives, we only need the square integral of f to be finite.

Theorem 6 (*L^2 Convergence of Fourier Series*)

If $\int_{-L}^{L} |f(x)|^2 \, dx < \infty$, then its full Fourier series converges to f in L^2.

A consequence of this result can be used to compute infinite series.

Corollary 2 (*Parseval's Identity*)

If $\int_{-L}^{L} |f(x)|^2\,dx < \infty$, then

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2 = \frac{1}{L}\int_{-L}^{L} |f(x)|^2\,dx.$$

Proof. From the mean square derivation, the optimal choice of coefficients $c_n = \frac{\langle f, X_N\rangle}{\langle X_n, X_n\rangle}$

$$0 \le E_N(c_1,\dots,c_N) = \langle f, f\rangle - \sum_{n=1}^{N} \frac{\langle f, X_n\rangle^2}{\langle X_n, X_n\rangle} = \langle f, f\rangle - \sum_{n=1}^{N} c_n^2 \langle X_n, X_n\rangle.$$

If f converges in L^2 then $E_N \to 0$. This implies that for

$$\sum_{n=1}^{N} c_n^2 \langle X_n, X_n\rangle = \sum_{n=1}^{N} \frac{\langle f, X_N\rangle^2}{\langle X_n, X_n\rangle} \to \langle f, f\rangle.$$

Because $\int_{-L}^{L} |f(x)|^2\,dx < \infty$, the full Fourier series converges in L^2. Therefore, applying this general result to the eigenfunctions corresponding to the full Fourier series implies

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2 = \frac{1}{L}\int_{-L}^{L} |f(x)|^2\,dx$$

since $\langle X_n, X_n\rangle = \langle Y_n, Y_n\rangle = L$ for $n \ge 1$ and $\langle X_0, X_0\rangle = \frac{L}{2}$. $\square$

2.2.4 Term by Term Manipulations of Fourier Series

Fourier series are nicer than general infinite series, so we can do manipulations of the series term by term in much more general settings. We state some easy to check sufficient conditions.

Theorem 7 (*Linearity of Fourier Series*)

If f_1, f_2 and f_1', f_2' are piecewise continuous on $[-L, L]$ then the full Fourier series of $c_1 f_1(x) + c_2 f_2(x)$ is the corresponding linear combination of the full Fourier series for f_1 and f_2.

Theorem 8 (*Term by Term Integration of Fourier Series*)

If f and f' are piecewise continuous on $[-L, L]$ then its full Fourier series can be integrated term by term to give a series that converges pointwise to the integral of f.

Remark 6. The series we get from term by term integration might not necessarily be a Fourier series because of the $\frac{a_0 x}{2}$ term. We can use the Fourier series of x and linearity to write it as a Fourier series.

Theorem 9 (*Term by Term Differentiation of Fourier Series*)

If f is continuous on $[-L, L]$, f' is piecewise continuous on $[-L, L]$, f'' is piecewise continuous on $[-L, L]$ and $f(-L) = f(L)$ then its full Fourier series can be differentiated term by term to give a series that converges to f' pointwise where f'' exists.

Remark 7. The conditions on term by term differentiation imply that f' has a pointwise convergent Fourier series. If we compute the Fourier coefficients of f', then under the continuity assumption on f, we will see that the Fourier series of f' is

$$f'(x) = \sum_{n=1}^{\infty} \frac{n\pi}{L}\Big(-a_n \sin\Big(\frac{n\pi x}{L}\Big) + b_n \cos\Big(\frac{n\pi x}{L}\Big)\Big)$$

where a_n and b_n are the corresponding Fourier coefficients of f.

2.3 Convergence of General Fourier Series

Consider the eigenvalue problem

$$-X''(x) = \lambda X(x) \qquad a < x < b \tag{8}$$

with symmetric boundary conditions. We will show that the eigenfunctions of (8) form an orthogonal basis for We have the following fact about its eigenvalues and eigenfunctions.

Theorem 10 (*Eigenvalues and Eigenfunctions*)

1. *The eigenvalues (λ_n) of (8) are real and they form an infinite sequence such that $\lambda_n \to \infty$.*
2. *The eigenfunctions (X_n) corresponding to distinct eigenvalues are orthogonal with respect to the $L^2([a, b])$ inner product.*

This implies that we have a sequence an infinite sequence of orthogonal eigenfunctions (X_n), so we can define the general Fourier series

$$f(x) = \sum_{n=1}^{\infty} c_n \langle f, X_n \rangle \qquad \text{where} \qquad c_n = \frac{\langle f, X_n \rangle}{\langle X_n, X_n \rangle}. \tag{9}$$

The general Fourier series also converge to the original function under some stronger assumptions than the ones for full Fourier series.

Theorem 11 (*Uniform Convergence of General Fourier Series*)

If $f \in C^2([a, b])$ and f satisfies the given symmetric boundary conditions, then the general Fourier series (9) converges uniformly to f on $[a, b]$.

The next result implies that the eigenfunctions form a orthogonal basis for L^2 functions.

Theorem 12 (*L^2 Convergence of General Fourier Series*)

If $f \in L^2([a, b])$, then the general Fourier series (9) converges to f in L^2 on $[a, b]$.

This Theorem actually implies a stronger result. If we adapt the proof of Corollary 2 then we also that (X_n) satisfies the general Parseval's identity

$$\sum_{n=1}^{\infty} c_n^2 \langle X_n, X_n \rangle = \langle f, f \rangle. \tag{10}$$

This implies that the eigenfunctions form a complete orthogonal basis for L^2 functions. These notions of convergence confirms the validity of separation of variables for a large class of boundary value problems we introduce in the Week 9 lecture summaries.

2.4 Example Problems

Problem 2.1. ($\star$) Without computing Fourier series of the following functions on $[-\pi, \pi]$, determine whether the Fourier series converges pointwise to $f(x)$ on $(-\pi, \pi)$ or uniformly to $f(x)$ on $[-\pi, \pi]$.

$$(a) \quad f(x) = \begin{cases} 2 & -\pi \le x < 0 \\ \frac{3}{2} & x = 0 \\ \cos x & 0 < x \le \pi \end{cases} \qquad\qquad (b) \quad f(x) = x^2 + x$$

Solution 2.1. Recall that f_N denotes the Nth partial sum of the full Fourier series of f.

Part (a)

1. *Pointwise Convergence:* The Fourier series clearly converges pointwise whenever $f(x)$ is continuous by the pointwise convergence theorem. We now examine the behavior at the point of discontinuity, $x = 0$. By the pointwise convergence theorem, we have

$$\lim_{N\to\infty} f_N(0) = \frac{f_{ext}(0^+) + f_{ext}(0^-)}{2} = \frac{2+1}{2} = \frac{3}{2}.$$

   In particular, since $f(0) = \frac{3}{2}$ we have

$$\lim_{N\to\infty} |f_N(0) - f(0)| = \Big| \lim_{N\to\infty} f_N(0) - f(0) \Big| = 0,$$

   so the Fourier series converges pointwise on $(-\pi, \pi)$.

2. *Uniform Convergence:* f is continuous, f' is piecewise continuous, but f does not satisfy the periodic boundary conditions $f(-\pi) = 2 \neq -1 = f(\pi)$. This is not enough to show that the series does not uniformly converge because the statement of the uniform convergence theorem only goes in one direction.

   To see the series does not converge uniformly we first check if the series converges at the endpoints. By the pointwise convergence theorem applied at the point π,

$$f_N(\pi) = \frac{f_{ext}(\pi^+) + f_{ext}(\pi^-)}{2} = \frac{f(-\pi) + f(\pi)}{2} = \frac{2-1}{2} = \frac{1}{2}$$

   and therefore,

$$\lim_{N\to\infty} |f_N(\pi) - f(\pi)| = \Big| \lim_{N\to\infty} f_N(\pi) - f(\pi) \Big| = \frac{1}{2} \geq 0.$$

   Since the series does not converge at one of the endpoints, we have

$$\lim_{N\to\infty} \|f_N(x) - f(x)\|_\infty := \lim_{N\to\infty} \sup_{x\in[-\pi,\pi]} |f_N(x) - f(x)| \geq \lim_{N\to\infty} |f_N(\pi) - f(\pi)| = \frac{3}{2} \neq 0,$$

   so f_N does not converge uniformly to f on $[-\pi, \pi]$.

Part (b)

1. *Pointwise Convergence:* $f(x) = x^2 + x$ is continuous, so $f(x^+) = f(x^-)$ for all $x \in (-\pi, \pi)$. By the pointwise convergence theorem, for all $x \in (-\pi, \pi)$ we have

$$\lim_{N\to\infty} f_N(x) = \frac{f_{ext}(x^+) + f_{ext}(x^-)}{2} = f(x).$$

   Therefore, f_N converges pointwise on $(-\pi, \pi)$.

2. *Uniform Convergence:* f is continuous, f' is piecewise continuous, but f does not satisfy the periodic boundary conditions $f(-\pi) = \pi^2 - \pi \neq \pi^2 + \pi = f(\pi)$. This is not enough to show that the series does not uniformly converge because the statement of the uniform convergence theorem only goes in one direction.

   To see the series does not converge uniformly we first check if the series converges at the endpoints. By the pointwise convergence theorem applied at the point π,

$$f_N(\pi) = \frac{f_{ext}(\pi^+) + f_{ext}(\pi^-)}{2} = \frac{f(-\pi) + f(\pi)}{2} = \frac{\pi^2 + \pi + \pi^2 - \pi}{2} = \pi^2$$

and therefore,

$$\lim_{N\to\infty} |f_N(\pi) - f(\pi)| \geq \left| \lim_{N\to\infty} f(\pi) - f_N(\pi) \right| = \pi^2 + \pi - \pi^2 = \pi.$$

Since the series does not converge at one of the endpoints, we have

$$\lim_{N\to\infty} \|f_N(x) - f(x)\|_\infty := \lim_{N\to\infty} \sup_{x\in[-\pi,\pi]} |f_N(x) - f(x)| \geq \lim_{N\to\infty} |f_N(\pi) - f(\pi)| \geq \pi \neq 0,$$

so f_N does not converge uniformly to f on $[-\pi,\pi]$.

Problem 2.2. ($\star$) Find the Fourier sine series of $f(x) = x$ on the interval $(0,\pi)$. Show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Solution 2.2. The sine series is the Fourier series of the odd extension of $f(x) = x$ to $-\pi \leq x \leq \pi$. We just compute the coefficients

a_0: Since $f_{ext}(x) = x$ is odd, the coefficient is given by

$$a_0 = 0.$$

a_n: Since $f_{ext}(x)\cos(nx) = x\cos(nx)$ is odd, the coefficient is given by

$$a_n = 0.$$

b_n: Since $f_{ext}(x)\sin(nx) = x\sin(nx)$ is even, the coefficient is given by

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^\pi x\sin(nx)\,dx = 2\cdot\frac{\sin(\pi n) - \pi n\cos(\pi n)}{\pi n^2} = \frac{2(-1)^{n+1}}{n}.$$

The corresponding Fourier sine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n\cos(nx) + b_n\sin(nx)\Big) = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\cdot\sin(nx).$$

We now apply Parseval's identity. Notice that for the odd extension of x to $[-\pi,\pi]$,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f_{ext}(x)^2\,dx = \frac{2}{\pi}\int_0^\pi x^2\,dx = \frac{2\pi^2}{3}.$$

Therefore, by Parseval's identity on the sine series, we have

$$\frac{2\pi^2}{3} = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \sum_{n=1}^{\infty}\frac{4}{n^2}$$

which implies

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Problem 2.3. ($\star\star$)

(a) Find Fourier series of $f(x) = \begin{cases} 0 & -2 \le x < 0 \\ 2-x & 0 \le x \le 2 \end{cases}$ and plot the extension of this function to the whole line.

(b) Determine whether the Fourier series converges pointwise to $f(x)$ on $(-2,2)$ and uniformly to $f(x)$ on $[-2,2]$.

(c) Compute

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

Solution 2.3.

(Part a) The Fourier series is given by

a_0**:** The coefficient is given by

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\int_{0}^{2} 2-x\,dx = 1.$$

a_n**:** Using integration by parts, the coefficient is given by

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\Big(\frac{n\pi x}{2}\Big)\,dx = \frac{1}{2}\int_{0}^{2}(2-x)\cos\Big(\frac{n\pi x}{2}\Big)\,dx = \frac{2-2\cos(\pi n)}{\pi^2 n^2} = \frac{2-2(-1)^n}{\pi^2 n^2}.$$

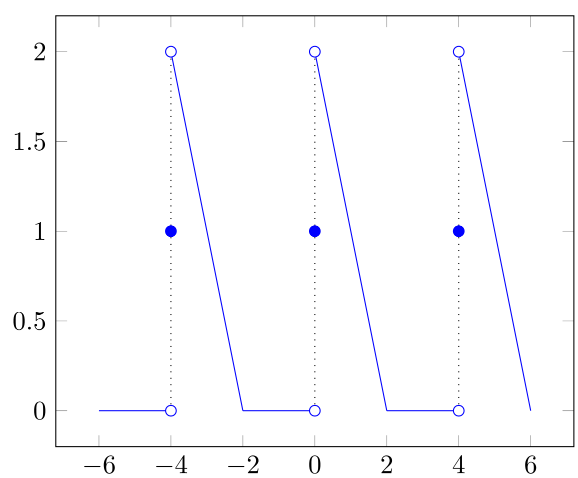
b_n**:** Using integration by parts, the coefficient is given by

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\Big(\frac{n\pi x}{2}\Big)\,dx = \frac{1}{2}\int_{0}^{2}(2-x)\sin\Big(\frac{n\pi x}{2}\Big)\,dx = \frac{2\pi n - 2\sin(\pi n)}{\pi^2 n^2} = \frac{2}{\pi n}.$$

The corresponding Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n\cos\Big(\frac{n\pi x}{L}\Big) + b_n\sin\Big(\frac{n\pi x}{L}\Big)\Big) = \frac{1}{2} + \sum_{n=1}^{\infty}\Big(\frac{2-2(-1)^n}{\pi^2 n^2}\cdot\cos\Big(\frac{n\pi x}{2}\Big) + \frac{2}{\pi n}\cdot\sin\Big(\frac{n\pi x}{2}\Big)\Big).$$

By the pointwise convergence theorem, the series converges pointwise to the average of the left and right endpoints of the periodic extension of $f(x)$ to $\mathbb{R}$. The plot of the Fourier series is given below:



(Part b) We now examine convergence.

1. Pointwise Convergence: By the pointwise convergence theorem, we have

$$\lim_{N\to\infty} f_N(0) = \frac{f_{ext}(0^+) + f_{ext}(0^-)}{2} = \frac{2+0}{2} = 1.$$

   In particular, since $f(0) = 2$ we have

$$\lim_{N\to\infty} |f_N(0) - f(0)| = \Big| \lim_{N\to\infty} f_N(0) - f(2) \Big| = 1 \neq 0,$$

   so the Fourier series does not converge pointwise on $(-2, 2)$.

2. Uniform Convergence: Uniform convergence is a stronger condition than pointwise convergence, that is, if $f_N(x)$ converges uniformly on $[-2, 2]$ then $f_N(x)$ converges pointwise on $(-2, 2)$. In particular, since $f_N(x)$ does not converge pointwise at $x = 0$, we have the Fourier series does not converge uniformly on $[-2, 2]$.

(Part c) We use the pointwise convergence theorem to compute the required series. At $x = 0$, we have

$$\lim_{N\to\infty} f_N(0) = \frac{f_{ext}(0^+) + f_{ext}(0^-)}{2} = \frac{2+0}{2} = 1.$$

Now evaluating the series at $x = 0$ gives

$$\begin{aligned}\frac{1}{2} + \sum_{n=1}^{\infty} \bigg(\frac{2 - 2(-1)^n}{\pi^2 n^2} \cdot \cos\Big(\frac{n\pi x}{2}\Big) + \frac{2}{\pi n} \cdot \sin\Big(\frac{n\pi x}{2}\Big)\bigg)\Big|_{x=0} &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2 - 2(-1)^n}{\pi^2 n^2} \\ &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{4}{\pi^2 (2n-1)^2}.\end{aligned}$$

Therefore, by the pointwise convergence theorem at $x = 0$, we have

$$\frac{1}{2} + \sum_{n=1}^{\infty} \frac{4}{\pi^2(2n-1)^2} = 1 \implies \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

Problem 2.4. ($\star\star$)

(a) Find the Fourier sine and cosine series of $f(x) = x(\pi - x)$ on $0 \le x \le \pi$.

(b) Show that

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}.$$

(c) Show that

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

Solution 2.4.

(Part a Cosine Series:) The cosine series is the Fourier series of the even extension of $f(x)$ to $-\pi \le x \le \pi$. We compute the coefficients

a_0**:** Since $f_{ext}(x)$ is even, the coefficient is given by

$$a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx = \frac{2}{\pi}\int_0^\pi x(\pi - x)\,dx = \frac{\pi^2}{3}.$$

a_n**:** Since $f_{ext}(x)\cos(nx)$ is even, the coefficient is given by

$$\begin{aligned} a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^\pi x(\pi - x)\cos(nx)\,dx &= -2\cdot\frac{\pi n - 2\sin(\pi n) + \pi n\cos(\pi n)}{\pi n^3} \\ &= -2\cdot\frac{1+(-1)^n}{n^2} \\ &= \begin{cases} 0 & n \text{ is odd} \\ -\frac{4}{n^2} & n \text{ is even} \end{cases}. \end{aligned}$$

b_n**:** Since $f_{ext}(x)\sin(nx)$ is odd, the coefficient is given by

$$b_n = 0.$$

The corresponding Fourier cosine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^\infty \Big(a_n\cos(nx) + b_n\sin(nx)\Big) = \frac{\pi^2}{6} - \sum_{n=1}^\infty \frac{1}{n^2}\cdot\cos(2nx).$$

(Part a Sine Series:) The sine series is the Fourier series of the odd extension of $f(x)$ to $-\pi \le x \le \pi$. We compute the coefficients

a_0**:** Since $f_{ext}(x)$ is odd, the coefficient is given by

$$a_0 = 0.$$

a_n**:** Since $f_{ext}(x)\cos(nx)$ is odd, the coefficient is given by

$$a_n = 0.$$

b_n**:** Since $f_{ext}(x)\sin(nx)$ is even, the coefficient is given by

$$\begin{aligned} b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^\pi x(\pi - x)\sin(nx)\,dx &= 2\cdot\frac{2 - \pi n\sin(\pi n) - 2\cos(\pi n)}{\pi n^3} \\ &= 4\cdot\frac{1-(-1)^n}{\pi n^3} \\ &= \begin{cases} \frac{8}{\pi n^3} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}. \end{aligned}$$

The corresponding Fourier sine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^\infty \Big(a_n\cos(nx) + b_n\sin(nx)\Big) = \sum_{n=1}^\infty \frac{8}{\pi(2n-1)^3}\cdot\sin((2n-1)x).$$

(Part b) We use the fact $\sin(\frac{(2n-1)\pi}{2}) = (-1)^{n+1}$ to compute the series. $f(x)$ is continuous at $\frac{\pi}{2}$, so the pointwise convergence theorem implies

$$f(\pi/2) = \frac{\pi^2}{4} = \sum_{n=1}^\infty \frac{8}{\pi(2n-1)^3}\cdot\sin((2n-1)x)\Big|_{x=\frac{\pi}{2}} = \sum_{n=1}^\infty \frac{8}{\pi(2n-1)^3}(-1)^{n+1}.$$

Rearranging terms implies

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}.$$

(Part c) To compute the other series, we use Parseval's identity. Notice that for the odd extension of $x(\pi - x)$ to $[-\pi, \pi]$,

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f_{ext}(x)^2 \, dx = \frac{2}{\pi} \int_0^{\pi} (x(\pi - x))^2 \, dx = \frac{\pi^4}{15}.$$

Therefore, by Parseval's identity on the sine series, we have

$$\frac{\pi^4}{15} = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \sum_{n=1}^{\infty} \frac{64}{\pi^2 (2n-1)^6}$$

which implies

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = \frac{\pi^6}{960}.$$

To recover the series in the question, we split the sum into its odd and even components,

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} + \sum_{n=1}^{\infty} \frac{1}{(2n)^6} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} + \frac{1}{64} \sum_{n=1}^{\infty} \frac{1}{n^6}.$$

Rearranging terms and using our formula for the sum of odd terms implies

$$\left(1 - \frac{1}{64}\right) \sum_{n=1}^{\infty} \frac{1}{n^6} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = \frac{\pi^6}{960} \implies \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{960} \cdot \frac{64}{63} = \frac{\pi^6}{945}.$$