1 Eigenvalue Problems

We introduce a class of Sturm–Liouville eigenvalue problems. These problems will appear when we solve PDEs on the finite interval using *separation of variables.* Consider the second order ODE on $[a, b]$ subject to some boundary conditions

$$\begin{cases} -X''(x) = \lambda X(x) & a < x < b \\ a_1X(a) + b_1X(b) + c_1X'(a) + d_1X'(b) = 0 \\ a_2X(a) + b_2X(b) + c_2X'(a) + d_2X'(b) = 0 \end{cases} \tag{1}$$

where $a_1, \ldots, d_2 \in \mathbb{R}$. A non-trivial solution X to (1) is called an *eigenfunction*, and the corresponding value of λ is called an *eigenvalue*.

Remark 1. This terminology should remind you of a concept from linear algebra. Recall that the eigenvalues λ and eigenvectors $\vec{v} \neq 0$ of a matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ are solutions to

$$\boldsymbol{A}\vec{v} = \lambda\vec{v}.$$

Since we are in finite dimensions, there are at most n eigenvalues. If $\boldsymbol{A}$ is symmetric, then eigenvectors corresponding to distinct eigenvalues are orthogonal. We can think of $L = -\frac{d^2}{dx}$ as a linear operator on X. In this context, solutions to the ODE in (1) satisfy

$$LX = \lambda X.$$

In this "infinite" dimensional case, there are infinitely many eigenvalues.

1.1 Common Eigenvalue Problems

We summarize the eigenfunctions and eigenvalues of several common eigenvalue problems.

1. Dirichlet Boundary Conditions:

$$\begin{cases} -X''(x) = \lambda X(x) & 0 < x < L \\ X(0) = X(L) = 0 \end{cases} \tag{2}$$

Eigenvalues: $\lambda_n = (\frac{n\pi}{L})^2$ for $n \geq 1$
Eigenfunctions: $X_n = \sin(\frac{n\pi x}{L})$ for $n \geq 1$

2. Neumann Boundary Conditions:

$$\begin{cases} -X''(x) = \lambda X(x) & 0 < x < L \\ X'(0) = X'(L) = 0 \end{cases} \tag{3}$$

Eigenvalues: $\lambda_n = (\frac{n\pi}{L})^2$ for $n \geq 0$
Eigenfunctions: $X_n = \cos(\frac{n\pi x}{L})$ for $n \geq 1$ and $X_0 = \frac{1}{2}$ for $n = 0$.

3. Periodic Boundary Conditions:

$$\begin{cases} -X''(x) = \lambda X(x) & -L < x < L \\ X(-L) - X(L) = X'(-L) - X'(L) = 0 \end{cases} \tag{4}$$

Eigenvalues: $\lambda_n = (\frac{n\pi}{L})^2$ for $n \geq 0$
Eigenfunctions: $X_n = \cos(\frac{n\pi x}{L})$ and $Y_n = \sin(\frac{n\pi x}{L})$ for $n \geq 1$ and $X_0 = \frac{1}{2}$ for $n = 0$.

4. Dirichlet–Neumann Mixed Boundary Conditions:

$$\begin{cases} -X''(x) = \lambda X(x) & 0 < x < L \\ X(0) = X'(L) = 0 \end{cases} \tag{5}$$

Eigenvalues: $\lambda_n = (\frac{(2n-1)\pi}{2L})^2$ for $n \geq 1$

Eigenfunctions: $X_n = \sin(\frac{(2n-1)\pi x}{2L})$ for $n \geq 1$.

5. Neumann–Dirichlet Mixed Boundary Conditions:

$$\begin{cases} -X''(x) = \lambda X(x) & 0 < x < L \\ X'(0) = X(L) = 0 \end{cases} \tag{6}$$

Eigenvalues: $\lambda_n = (\frac{(2n-1)\pi}{2L})^2$ for $n \geq 1$

Eigenfunctions: $X_n = \cos(\frac{(2n-1)\pi x}{2L})$ for $n \geq 1$.

Remark 2. Notice that if X is an eigenfunction of (1), then cX is also an eigenfunction for any number $c \neq 0$. This means that the eigenfunctions in the table are unique up to a scaling factor.

1.2 Orthogonality of Eigenfunctions

Definition 1. Consider continuous functions f, g defined on $[a, b]$. The L^2-*inner product* of these functions are given by

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,dx.$$

We say that the functions f and g are *orthogonal* if

$$\langle f, g \rangle = 0.$$

Definition 2. The boundary conditions of (1) are *symmetric* if

$$\big(f'(x)g(x) - f(x)g'(x)\big)\Big|_{x=a}^{x=b} = f'(b)g(b) - f(b)g'(b) - f'(a)g(a) + f(a)g'(a) = 0. \tag{7}$$

for functions f and g that solve (1). All the standard eigenvalue problems we encounter in this course will have symmetric boundary conditions.

Theorem 1 (*Orthogonality of Eigenfunctions*)

If the eigenvalue problem (1) has symmetric boundary conditions, then the eigenfunctions corresponding to distinct eigenvalues are orthogonal.

Proof. Let X_1 and X_2 be distinct solutions to (1), that is for $\lambda_1 \neq \lambda_2$,

$$-X_1'' = \lambda_1 X_1 \qquad \text{and} \qquad -X_2'' = \lambda_2 X_2.$$

We can check orthogonality directly,

$$(\lambda_2 - \lambda_1)\langle X_1, X_2 \rangle = (\lambda_2 - \lambda_1)\int_a^b X_1(x)X_2(x)\,dx = \int_a^b X_1''(x)X_2(x) - X_1(x)X_2''(x)\,dx.$$

Integrating by parts implies

$$\int_a^b X_1''(x)X_2(x) - X_1(x)X_2''(x)\,dx = \big(X_1'(x)X_2(x) - X_1(x)X_2'(x)\big)\Big|_{x=a}^{x=b} = 0$$

because X_1 and X_2 satisfy the symmetric boundary condition (7). Since $\lambda_1 - \lambda_2 \neq 0$, $\langle X_1, X_2 \rangle = 0$ so X_1 and X_2 are orthogonal. □

Remark 3. We can have distinct eigenfunctions for repeated eigenvalue. They might not be orthogonal, but we can use the Gram–Schmidt process extract a orthogonal set.

1.3 Example Problems

Constant Coefficient Second Order ODE: Recall that an ODE of the form

$$ay'' + by' + cy = 0$$

is a homogeneous second order constant coefficient ODE. The ODE is solved by finding the roots r_1 and r_2 of the characteristic polynomial

$$C(r) = ar^2 + br + c = 0.$$

The general form of the solution is given by

$$y(x) = y(x) = \begin{cases} C_1 e^{r_1 x} + C_2 e^{r_2 x} & r_1, r_2 \in \mathbb{R}, r_1 \neq r_2 \\ C_1 e^{rx} + C_2 x e^{rx} & r_1 = r_2 = r \in \mathbb{R} \\ C_1 e^{\alpha x}\cos(\beta x) + C_2 e^{\alpha x}\sin(\beta x) & r_1 = \alpha + i\beta, r_2 = \alpha - i\beta, \beta \neq 0 \end{cases}.$$

Remark 4. In the case when $r_1 = -r_2 \in \mathbb{R}$, it will be convenient to write the solution in the form

$$y(x) = C_1 \cosh(r_1 x) + C_2 \sinh(r_1 x).$$

We can check that this form also gives us a pair of linearly independent solutions to the ODE.

Problem 1.1. ($\star$) Solve the eigenvalue problem

$$\begin{cases} -X''(x) = \lambda X(x) & 0 < x < L \\ X(0) = X(L) = 0 \end{cases}$$

Solution 1.1. The solution to the ODE $-X''(x) = \lambda X(x)$ has different forms depending on the signs of λ, so we must consider each case separately.

Case $\lambda = \beta^2 > 0$: For $\beta > 0$, we define $\lambda = \beta^2$. We first find the general solution to the ODE

$$-X''(x) = \beta^2 X(x) \implies X''(x) + \beta^2 X(x) = 0.$$

The corresponding characteristic polynomial roots are $r = \pm\beta i$, so

$$X(x) = A\cos(\beta x) + B\sin(\beta x).$$

We now solve for the values of β that give nontrivial solutions to the boundary conditions. Plugging the solution into the boundary conditions gives

$$\begin{aligned} A &= 0 \\ A\cos(\beta L) + B\sin(\beta L) &= 0. \end{aligned}$$

We can write this system of equations in matrix form

$$\begin{bmatrix} 1 & 0 \\ \cos(\beta L) & \sin(\beta L) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

which has a non-trivial solution when

$$\det\left(\begin{bmatrix} 1 & 0 \\ \cos(\beta L) & \sin(\beta L) \end{bmatrix}\right) = 0 \implies \sin(\beta L) = 0 \implies \beta = \frac{n\pi}{L}.$$

Since $\beta > 0$, we must take $n \geq 1$. To find the eigenfunction, we now substitute $\beta_n = \frac{n\pi}{L}$ for $n \geq 1$ back into the matrix

$$\begin{bmatrix} 1 & 0 \\ \cos\left(\frac{n\pi}{L}L\right) & \sin\left(\frac{n\pi}{L}L\right) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ (-1)^n & 0 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

to conclude that $A = 0$ and B can be arbitrary. Therefore, the positive eigenvalues and eigenfunctions are

$$\lambda_n = \beta_n^2 = \left(\frac{n\pi}{L}\right)^2 \qquad \text{and} \qquad X_n = \sin\left(\frac{n\pi}{L}x\right).$$

<u>Case $\lambda = 0$</u>: We first find the general solution to the ODE

$$-X''(x) = 0 \implies X = A + Bx.$$

The corresponding characteristic polynomial has repeated roots $r = 0$, so

$$X(x) = A + Bx.$$

Plugging the solution into the boundary conditions gives

$$\begin{aligned} A &= 0 \\ A + BL &= 0. \end{aligned}$$

We can write this system of equations in matrix form

$$\begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

which only has the trivial solution $A = B = 0$ because

$$\det\left(\begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix}\right) = L \neq 0.$$

Therefore, $X_0(x) = 0$ is the only solution to the boundary value problem, so we have no zero eigenvalues.

<u>Case $\lambda = -\beta^2 < 0$</u>: For $\beta > 0$, we define $\lambda = -\beta^2$. We first find the general solution to the ODE

$$-X''(x) = -\beta^2 X(x) \implies X''(x) - \beta^2 X(x) = 0.$$

The corresponding characteristic polynomial roots are $r = \pm\beta$, so

$$X(x) = A\cosh(\beta x) + B\sinh(\beta x).$$

We now solve for the values of β that give nontrivial solutions to the boundary conditions. Plugging the solution into the boundary conditions gives

$$\begin{aligned} A &= 0 \\ A\cosh(\beta L) + B\sinh(\beta L) &= 0. \end{aligned}$$

We can write this system of equations in matrix form

$$\begin{bmatrix} 1 & 0 \\ \cosh(\beta L) & \sinh(\beta L) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

which has a non-trivial solution when

$$\det\left(\begin{bmatrix} 1 & 0 \\ \cosh(\beta L) & \sinh(\beta L) \end{bmatrix}\right) = 0 \implies \sinh(\beta L) = 0 \implies \beta = 0.$$

Since $\beta > 0$, there are no choices of β that result in a non-trivial solution for A, B. We can conclude that there are no negative eigenvalues.

Summary: We have shown that the eigenvalues and eigenfunctions corresponding to Dirichlet boundary conditions are

Eigenvalues: $\lambda_n = (\frac{n\pi}{L})^2$ for $n \geq 1$

Eigenfunctions: $X_n = \sin(\frac{n\pi x}{L})$ for $n \geq 1$

Remark 5. We could have defined $X_n = B\sin(\frac{n\pi}{L}x)$ for any $B \neq 0$ to be an eigenfunction since all constant multiples of an eigenfunctions are eigenfunctions. It is standard to choose $B = 1$.

Problem 1.2. ($\star$) Solve the eigenvalue problem

$$\begin{cases} -X''(x) = \lambda X(x) & 0 < x < L \\ X'(0) = X'(L) = 0 \end{cases}$$

Solution 1.2. The solution to the ODE $-X''(x) = \lambda X(x)$ has different forms depending on the signs of λ, so we must consider each case separately.

Case $\lambda = \beta^2 > 0$: For $\beta > 0$, we define $\lambda = \beta^2$. We first find the general solution to the ODE

$$-X''(x) = \beta^2 X(x) \implies X''(x) + \beta^2 X(x) = 0.$$

The corresponding characteristic polynomial roots are $r = \pm\beta i$, so

$$X(x) = A\cos(\beta x) + B\sin(\beta x).$$

We now solve for the values of β that give nontrivial solutions to the boundary conditions. Plugging the solution into the boundary conditions gives

$$\begin{aligned} \beta B &= 0 \\ -A\beta\sin(\beta L) + B\beta\cos(\beta L) &= 0. \end{aligned}$$

We can write this system of equations in matrix form

$$\begin{bmatrix} 0 & \beta \\ -\beta\sin(\beta L) & \beta\cos(\beta L) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

which has a non-trivial solution when

$$\det\left(\begin{bmatrix} 0 & \beta \\ -\beta\sin(\beta L) & \beta\cos(\beta L) \end{bmatrix}\right) = 0 \implies \beta^2\sin(\beta L) = 0 \implies \beta = \frac{n\pi}{L}.$$

Since $\beta > 0$, we must take $n \geq 1$. To find the eigenfunction, we now substitute $\beta_n = \frac{n\pi}{L}$ for $n \geq 1$ back into the matrix

$$\begin{bmatrix} 0 & \frac{n\pi}{L} \\ -\frac{n\pi}{L}\sin\left(\frac{n\pi}{L}L\right) & \frac{n\pi}{L}\cos\left(\frac{n\pi}{L}L\right) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 & \frac{n\pi}{L} \\ 0 & \frac{n\pi}{L}(-1)^n \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

to conclude that $B = 0$ and A can be arbitrary. Therefore, the positive eigenvalues and eigenfunctions are

$$\lambda_n = \beta_n^2 = \left(\frac{n\pi}{L}\right)^2 \qquad \text{and} \qquad X_n = \cos\left(\frac{n\pi}{L}x\right).$$

Case $\lambda = 0$: We first find the general solution to the ODE

$$-X''(x) = 0 \implies X = A + Bx.$$

The corresponding characteristic polynomial has repeated roots $r = 0$, so

$$X(x) = A + Bx.$$

Plugging the solution into the boundary conditions gives

$$\begin{aligned} B &= 0 \\ B &= 0. \end{aligned}$$

We can write this system of equations in matrix form

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

to conclude that $B = 0$ and A can be arbitrary. Therefore, $X_0(x) = \frac{1}{2}$ is the eigenfunction corresponding to the zero eigenvalue.

Case $\lambda = -\beta^2 < 0$: For $\beta > 0$, we define $\lambda = -\beta^2$. We first find the general solution to the ODE

$$-X''(x) = -\beta^2 X(x) \implies X''(x) - \beta^2 X(x) = 0.$$

The corresponding characteristic polynomial roots are $r = \pm\beta$, so

$$X(x) = A\cosh(\beta x) + B\sinh(\beta x).$$

We now solve for the values of β that give nontrivial solutions to the boundary conditions. Plugging the solution into the boundary conditions gives

$$\begin{aligned} B\beta &= 0 \\ A\beta\sinh(\beta L) + B\beta\cosh(\beta L) &= 0. \end{aligned}$$

We can write this system of equations in matrix form

$$\begin{bmatrix} 0 & \beta \\ \beta\sinh(\beta L) & \beta\cosh(\beta L) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

which has a non-trivial solution when

$$\det\left(\begin{bmatrix} 0 & \beta \\ \beta\sinh(\beta L) & \beta\cosh(\beta L) \end{bmatrix}\right) = 0 \implies -\beta^2\sinh(\beta L) = 0 \implies \beta = 0.$$

Since $\beta > 0$, there are no choices of β that result in a non-trivial solution for A, B. We can conclude that there are no negative eigenvalues.

Summary: We have shown that the eigenvalues and eigenfunctions corresponding to Neumann boundary conditions are

Eigenvalues: $\lambda_n = (\frac{n\pi}{L})^2$ for $n \geq 0$

Eigenfunctions: $X_n = \cos(\frac{n\pi x}{L})$ for $n \geq 1$ and $X_0 = \frac{1}{2}$ for $n = 0$.

Remark 6. We chose the zero eigenfunction to be $X_0 = \frac{1}{2}$ because it matches the convention used for coefficients of the Fourier cosine series.

Problem 1.3. ($\star\star$) Find the positive eigenvalues and eigenfunctions of

$$\begin{cases} X^{(4)} = \lambda X & 0 < x < L \\ X(0) = X(L) = X''(0) = X''(L) = 0 \end{cases}$$

Solution 1.3. We want to find non-trivial solutions corresponding to $\lambda > 0$.

General Solution: We are interested in positive eigenvalues, so we can set $\lambda = \beta^4 > 0$, where $\beta > 0$. We first find the general solution to the ODE

$$X^{(4)} = \beta^4 X \qquad 0 < x < L.$$

This is a fourth order constant coefficient ODE with characteristic polynomial roots $r = \pm\beta, \pm\beta i$, which corresponds to the solution

$$X(x) = A\cos(\beta x) + B\sin(\beta x) + C\cosh(\beta x) + D\sinh(\beta x).$$

Particular Solution: We now solve for the values of β that give nontrivial solutions to the boundary conditions. Plugging the solution into the boundary conditions gives

$$\begin{aligned} A + C &= 0 \\ A\cos(\beta L) + B\sin(\beta L) + C\cosh(\beta L) + D\sinh(\beta L) &= 0 \\ -\beta^2 A + \beta^2 C &= 0 \\ -A\beta^2\cos(\beta L) - B\beta^2\sin(\beta L) + C\beta^2\cosh(\beta L) + D\beta^2\sinh(\beta L) &= 0. \end{aligned}$$

Since $\beta > 0$, the first and third equation implies $A + C = 0$ and $-A + C = 0$ which can only happen when $A = C = 0$. We now have to solve the system

$$\begin{aligned} B\sin(\beta L) + D\sinh(\beta L) &= 0 \\ -B\beta^2\sin(\beta L) + D\beta^2\sinh(\beta L) &= 0. \end{aligned}$$

We can write this system of equations in matrix form

$$\begin{bmatrix} \sin(\beta L) & \sinh(\beta L) \\ -\beta^2\sin(\beta L) & \beta^2\sinh(\beta L) \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

which has a non-trivial solution when

$$\det\left(\begin{bmatrix} \sin(\beta L) & \sinh(\beta L) \\ -\beta^2\sin(\beta L) & \beta^2\sinh(\beta L) \end{bmatrix} \right) = 0 \implies 2\beta^2\sinh(\beta L)\sin(\beta L) = 0.$$

Since $\beta > 0$ and $\sinh(\beta L) > 0$, this simplifies to $\sin(\beta L) = 0$, which occurs precisely when

$$\beta_n = \frac{n\pi}{L}, \quad n \geq 1.$$

Therefore, the corresponding eigenvalues are $\lambda_n = \frac{n^4\pi^4}{L^4}$. Furthermore, notice that from the equation

$$B\sin(\beta_n L) + D\sinh(\beta_n L) = 0,$$

we must have $D = 0$ since $\sinh(\beta_n L) > 0$ and $\sin(\beta_n L) = 0$. Finally, for $n \geq 1$, the corresponding eigenfunction (taking $B = 1$) for the eigenvalue $\lambda_n = (\frac{n\pi}{L})^4$ is

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right).$$

Remark 7. It turns out that $X_n(x)$ form a basis of the continuous functions on $[0, L]$, so these are all the eigenfunctions.