1 The Transport Equation

The *transport equation* models the concentration of a substance flowing in a fluid at a constant rate.

Definition 1. For parameters $c \in \mathbb{R}$, the *transport equation* on $\mathbb{R} \times \mathbb{R}^+$ is

$$u_t + cu_x = 0. \tag{1}$$

The corresponding IVP for the transport equation is

$$\begin{cases} u_t + cu_x = 0 & x \in \mathbb{R}, t > 0 \\ u|_{t=0} = f(x) & x \in \mathbb{R}. \end{cases} \tag{2}$$

The solution to this equation is derived using a method called the *method of characteristics*.

Theorem 1 (*Solution to the Transport Equation*)

(a) *The general solution to* (1) *is*

$$u(x,t) = \phi(x - ct), \tag{3}$$

where ϕ is an arbitrary function.

(b) *The particular solution to* (2) *is*

$$u(x,t) = f(x - ct). \tag{4}$$

1.1 Derivation of the General Solution

We give two derivations of (3). Consider the general constant coefficient equation on $\mathbb{R}^2$

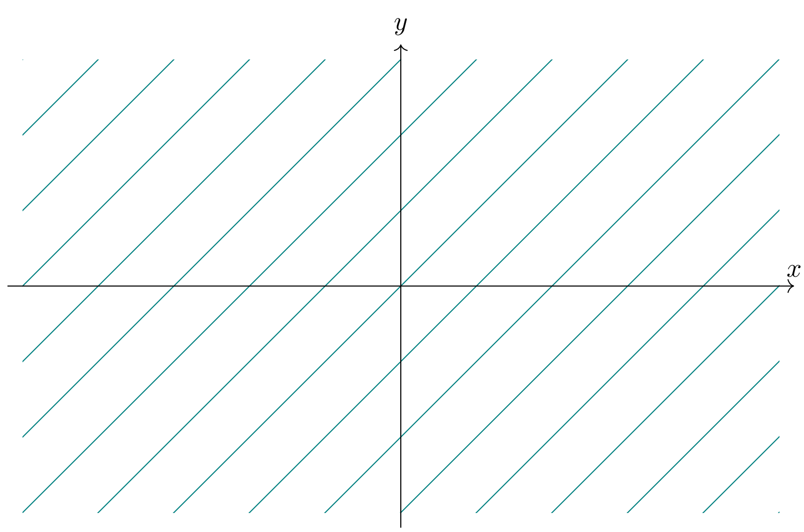
$$au_x + bu_y = 0. \tag{5}$$

1.1.1 Method 1: Integral Curves

We present a geometric derivation of general solution. If we define $\vec{c} = (a, b)$, then the (5) can be written as

$$\nabla_c u := \vec{c} \cdot \nabla u = au_x + bu_y = 0.$$

That is, the directional derivative of u in direction $\vec{c}$ is 0, so u is constant along the lines parallel to $\vec{c}$.



Notice that the vector $\vec{c}$ has corresponds to lines with slope $\frac{b}{a}$, so it described by the integral curves satisfying the *characteristic equations*

$$\frac{dy}{dx} = \frac{b}{a} \iff \frac{dx}{a} = \frac{dy}{b}. \tag{6}$$

The equations for lines with slope $\frac{b}{a}$ can be recovered by integrating (6),

$$\frac{dx}{a} = \frac{dy}{b} \implies ay = bx + C \implies ay - bx = C.$$

Therefore, the solution only depends on the family of *characteristic curves* of the form $ay - bx = C$. These lines can be parameterized by C, so

$$u(x, y) = \phi(C) = \phi(ay - bx)$$

for some function $\phi : \mathbb{R} \to \mathbb{R}$.

1.1.2 Method 2: Change of Variables

Using the characteristic equations

$$\frac{dx}{a} = \frac{dy}{b}, \tag{7}$$

we can do a change of variables to reduce the PDE into an ODE. Integrating (7) implies

$$\frac{dx}{a} = \frac{dy}{b} \implies ay = bx + C \implies ay - bx = C \implies y = \frac{C + bx}{a}.$$

We now treat y as a function of C and x and define

$$v(x, C) := u(x, y(C, x)) \quad \text{where} \quad y(C, x) := \frac{C + bx}{a}.$$

By the multivariable chain rule,

$$\frac{\partial}{\partial x} v(x, C) = u_x + u_y \cdot \frac{\partial}{\partial x} y(C, x) = u_x + u_y \cdot \frac{b}{a} = 0.$$

since u satisfies the equation (5). This is now an ODE in x, so we can integrate both sides with respect to x to conclude that the general solution is of the form

$$v(x, C) = \phi(C)$$

for some function $\phi : \mathbb{R} \to \mathbb{R}$. Since $C = ay - bx$ and $v(x, C) = u(x, y)$, we can write this equation back in terms of u to conclude

$$u(x, y) = v(x, ay - bx) = \phi(ay - bx).$$

Remark 1. We could've also written x as a function of y. If we did this, then we will have to treat x as a function of C and y,

$$x(C, y) = \frac{ay - C}{b}.$$

One can check that this choice of change of variables will give us the same solution.

Remark 2. One can check that Method 2 works even when $a = 0$ or $b = 0$ after a small modification.

1.2 Particular Solution

Suppose we now specify the boundary condition

$$\begin{cases} au_x + bu_y = 0 & x \in \mathbb{R}, y > 0 \\ u|_{y=0} = f(x) & x \in \mathbb{R}. \end{cases} \tag{8}$$

To recover the particular solution (4) from the general solution, we simply plug the general solution

$$u(x,y) = \phi(ay - bx)$$

into the boundary condition and solve for the yet to be determined function ϕ

$$u|_{y=0} = f(x) \implies u(x,0) = \phi(-bx) = f(x) \overset{s=-bx}{\implies} \phi(s) = f\Big(-\frac{s}{b}\Big) \quad \text{for } s \in \mathbb{R}.$$

Therefore,

$$u(x,y) = \phi(ay - bx) = f\Big(-\frac{ay - bx}{b}\Big) = f\Big(x - \frac{a}{b}y\Big).$$

1.3 Example Problems

Problem 1.1. ($\star$) Solve the boundary value problem

$$\begin{cases} u_x + xu_y = 0 & x \in \mathbb{R}, y \in \mathbb{R} \\ u|_{x=0} = \sin(y) & y \in \mathbb{R}. \end{cases} \tag{9}$$

In which region of the xy-plane is the solution uniquely determined by the initial condition?

Solution 1.1. We explicitly solve this first order linear PDE using a change of variables.

Characteristic Equations: From the characteristic equations

$$\frac{dx}{1} = \frac{dy}{x}$$

we get the family of characteristic curves

$$y = \frac{x^2}{2} + C.$$

General Solution: Using the change of variables $y(C,x) = \frac{x^2}{2} + C$ and $v(x,C) = u(x, y(C,x))$, we get

$$\frac{\partial}{\partial x} v(x,C) = u_x + u_y \cdot \frac{\partial}{dx} y(C,x) = u_x + u_y \cdot x = 0 \implies v = \phi(C),$$

for some yet to be determined $\phi : \mathbb{R} \to \mathbb{R}$. Since $C = y - \frac{x^2}{2}$, we can write this back into our original equation to conclude that

$$u(x,y) = \phi\Big(y - \frac{x^2}{2}\Big). \tag{10}$$

Particular Solution: Plugging in the general solution into the boundary condition, we see that

$$u|_{x=0} = \sin(y) \implies \phi(y) = \sin(y) \overset{y=s}{\implies} \phi(s) = \sin(s) \quad \text{for } s \in \mathbb{R},$$

so

$$u(x,y) = \sin\Big(y - \frac{x^2}{2}\Big).$$

Notice that the condition $u|_{x=0} = \sin(y)$ specifies $\phi(s)$ for all $s \in \mathbb{R}$ so this boundary condition uniquely determines $u(x,y)$ in the entire xy plane.

Problem 1.2. ($\star\star$) Consider the boundary value problem

$$\begin{cases} u_x + x u_y = 0 & x \in \mathbb{R}, y \in \mathbb{R} \\ u|_{y=0} = f(x) & x \in \mathbb{R}. \end{cases} \tag{11}$$

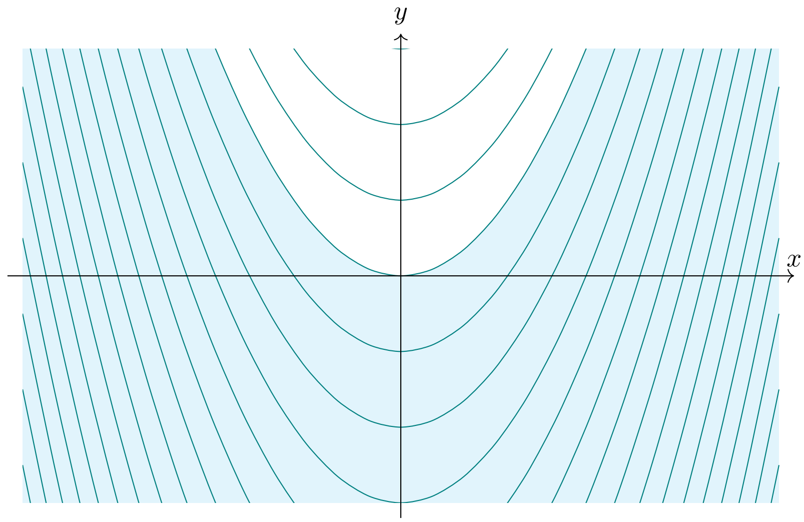
(a) What conditions do we need on f to ensure a solution exists?

(b) In which region of the xy-plane is the solution uniquely determined by the initial condition?

Solution 1.2. We use a geometric approach to determine if the BVP is well posed without solving it.

Characteristic Curves: At the point (x, y), the directional derivative in the direction $\vec{c} = (1, x)$ vanishes by (11). Since $\nabla_c u = 0$, u is constant along the characteristic curves satisfying

$$\frac{dx}{1} = \frac{dy}{x} \implies y = \frac{x^2}{2} + C.$$



(a) Since the solution is constant on the characteristic curves and the curves $y = \frac{x^2}{2} + C$ intersect the line $y = 0$ at the points $\pm\sqrt{-2C}$ for $C \leq 0$, we need $f(x)$ to be an even function for a solution to exist, i.e. $f(-\sqrt{-2C}) = f(\sqrt{-2C})$ for all $C \leq 0$ in addition to standard differentiability assumptions.

(b) Since the solution is constant on the characteristic curves and the curves $y = \frac{x^2}{2} + C$ intersect the line $y = 0$ only for $C \leq 0$, the solution is only uniquely determined on the shaded region. That is, on the points in the xy-plane (the shaded region in the picture above) such that

$$y - \frac{x^2}{2} \leq 0.$$

Remark 3. From Problem 1.1, we know the general solution is given by (10). Plugging this in the boundary conditions gives

$$u|_{y=0} = f(x) \implies \phi\Big(-\frac{x^2}{2}\Big) = f(x) \overset{s=-\frac{x^2}{2}}{\implies} \phi(s) = f(\pm\sqrt{-2s}) \quad \text{for } s \leq 0.$$

Since $f(\sqrt{2s}) = \phi(s) = f(-\sqrt{2s})$ for $s \geq 0$, we must have f is even. Furthermore, the initial condition is only specified for $s \leq 0$ so $u(x, y) = \phi(y - \frac{x^2}{2})$ is only uniquely determined for $y - \frac{x^2}{2} \leq 0$.

2 First Order Semilinear Equations

We want to find a formal solution to the first order semilinear PDEs of the form

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u). \tag{12}$$

The principles used to solve the transport equation can be extended to solve many first order semilinear equations. The change of variable computation in these general cases is almost identical to the one in Problem 1.1, so we can simplify the procedure by formally solving a system of characteristic equations.

Remark 4. Of course the methodology to solve semilinear equations will also apply to the simpler case of linear first order equations.

2.1 The Method of Characteristics

Using a change of variables corresponding to characteristic lines, we can reduce the problem to a system of 3 ODEs. The solution follows by simply solving two ODEs in the resulting system. This approach is called *the method of characteristics.*

Step 1: Formally, we want to solve the following system of equations

$$\frac{dx}{a(x,y)} = \frac{dy}{b(x,y)} = \frac{du}{c(x,y,u)}. \tag{13}$$

Step 2: We first find the characteristic curve by solving the first pair,

$$\frac{dx}{a(x,y)} = \frac{dy}{b(x,y)} \Leftrightarrow \frac{dy}{dx} = \frac{b(x,y)}{a(x,y)} \Leftrightarrow \frac{dx}{dy} = \frac{a(x,y)}{b(x,y)}.$$

We will get a characteristic line of the form $C = f(x,y)$.

Step 3: We now can now find the general solution by solving either

$$\frac{dx}{a(x,y)} = \frac{du}{c(x,y,u)} \qquad \text{or} \qquad \frac{dy}{b(x,y)} = \frac{du}{c(x,y,u)}.$$

We choose to solve the ODE that is easier to solve. We may need to use the characteristic curve and the implicit function theorem to write $f(x,y) = C$ as $y = y(C,x)$ or $x = x(C,y)$ to eliminate a variable to solve this ODE. When we solve the ODE, we must remember to write the constant of integration we get as a function $\phi(C) = \phi(f(x,y))$.

Step 4: If we are given some initial conditions, then we can find the specific form of ϕ.

Remark 5. This approach is a generalization of the formal computations used to solve separable ODEs. There is a hidden change of variables that allows us to manipulate the differentials in a rigorous way. An explanation of this method is outlined in Section 2.3.

Remark 6. Depending on which systems you solve, the general solution may appear to be different. However, since the solutions are defined in terms an arbitrary function ϕ, the different general solutions are equivalent. Solving for the particular solution will result in the same answer provided the PDE is well-posed.

Remark 7. This procedure can be adapted to solve problems in $\mathbb{R}^n$ as well. In $\mathbb{R}^3$, the main difference is that integral curves are now curves in $\mathbb{R}^3$, so they are parametrized by a two parameters instead of one in the $\mathbb{R}^2$ case. Some examples in $\mathbb{R}^3$ are explained in Problem 2.3 and Problem 2.4.

Remark 8. We can interpret the solution $z = u(x,y)$ as a surface in $\mathbb{R}^3$. Because $\vec{n} = (u_x, u_y, -1)$ defines the normal vectors to the surface $z = u(x,y)$, the PDE (12) says that the vector field $(a(x,y), b(x,y), c(x,y,z))$ is perpendicular to $\vec{n}$ at each point. This implies the integral curve equations (13) defines an integral surface in $\mathbb{R}^3$ that is tangent to the characteristic direction (a,b,c).

2.2 Example Problems

2.2.1 Semilinear Equations in $\mathbb{R}^2$

Problem 2.1. ($\star$) Find the general solutions to the following equations

$$-4u_x + u_y + u = 0, \tag{1}$$

$$-2u_x + 4u_y = e^{x+3y} - 5u. \tag{2}$$

Solution 2.1.

(1) We have the system of equations

$$\frac{dx}{-4} = \frac{dy}{1} = \frac{du}{-u}.$$

Characteristic Curve: We start by solving the equation involving the first and second term,

$$\frac{dx}{-4} = \frac{dy}{1} \Rightarrow \frac{dx}{dy} = -4 \Rightarrow C = x + 4y.$$

General Solution: We now solve the equation involving the second and third term,

$$\frac{dy}{1} = \frac{du}{-u} \Rightarrow \frac{du}{dy} = -u.$$

This is a separable ODE, which has solution

$$\log|u| = -y + \psi(C) \Rightarrow u = \pm e^{\psi(C)} e^{-y}.$$

Since $\psi(C)$ is an arbitrary function and $u \equiv 0$ is a solution, we might can redefine $\pm e^{\psi(C)} =: \phi(C)$. Since $C = x + 4y$, we have our general solution is

$$u(x,y) = \phi(x + 4y)e^{-y}$$

for some $\phi : \mathbb{R} \to \mathbb{R}$.

Remark 9. To find the general solution, if we solved the equation involving the first and third term

$$\frac{dx}{-4} = \frac{du}{-u},$$

we will get a general solution

$$u(x,y) = \psi(x + 4y)e^{\frac{x}{4}},$$

for some $\psi : \mathbb{R} \to \mathbb{R}$. This solution might look completely different, but it is actually of the same form. To see this, we can define $\psi(s) = e^{-\frac{s}{4}}\phi(s)$ to conclude that

$$u(x,y) = \psi(x + 4y)e^{\frac{x}{4}} = \phi(x + 4y)e^{-\frac{x+4y}{4}}e^{\frac{x}{4}} = \phi(x + 4y)e^{-y},$$

so the solutions are actually equivalent.

(2) We have the system of equations

$$\frac{dx}{-2} = \frac{dy}{4} = \frac{du}{e^{x+3y} - 5u}.$$

Characteristic Curve: We start by solving the equation involving the first and second term,

$$\frac{dx}{-2} = \frac{dy}{4} \Rightarrow \frac{dy}{dx} = -2 \Rightarrow C = y + 2x.$$

General Solution: We now solve the equation involving the first and third term,

$$\frac{dx}{-2} = \frac{du}{e^{x+3y} - 5u} \Rightarrow \frac{du}{dx} = -\frac{1}{2}(e^{x+3y} - 5u) \Rightarrow \frac{du}{dx} - \frac{5}{2}u = -\frac{1}{2}e^{x+3y}.$$

There is a y variable appearing in this ODE that we must eliminate first. Since $y = C - 2x$, we need to solve

$$\frac{du}{dx} - \frac{5}{2}u = -\frac{1}{2}e^{-5x+3C}.$$

This is a linear ODE, which can be solved using an integrating factor of the form $I(x) = e^{-\frac{5}{2}x}$, which gives us

$$u = e^{\frac{5}{2}x}\bigg(-\frac{1}{2}\int e^{-5x+3C}e^{-\frac{5}{2}x}\,dx\bigg) = -\frac{1}{2}e^{\frac{5}{2}x}\bigg(\frac{2e^{-\frac{15}{2}x+3C}}{-15} + \psi(C)\bigg) \Rightarrow u = \frac{1}{15}e^{-5x+3C} - \frac{1}{2}\psi(C)e^{\frac{5}{2}x}.$$

Since $C = y + 2x$, if we set $\phi(s) = -\frac{1}{2}\psi(s)$ then we get the general solution

$$u(x,y) = \frac{1}{15}e^{x+3y} + \phi(y+2x)e^{\frac{5}{2}x},$$

for some $\phi : \mathbb{R} \to \mathbb{R}$.

Problem 2.2. ($\star\star$) Find a solution to the initial value problem

$$xu_x + yu_y = xe^{-u}$$

with $u(x,y) = 0$ on $\{y = x^2\}$.

Solution 2.2. We have the system of equations

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{xe^{-u}}.$$

Characteristic Curves: We start by solving the equation involving the first and second term,

$$\frac{dx}{x} = \frac{dy}{y} \Rightarrow y = Cx.$$

General Solution: We now solve the equation involving first and third term,

$$\frac{dx}{x} = \frac{du}{xe^{-u}} \Rightarrow \frac{du}{dx} = e^{-u} \Rightarrow e^u = x + \phi(C).$$

Since $C = \frac{y}{x}$, we have our general solution is

$$u(x,y) = \ln\Big(x + \phi\Big(\frac{y}{x}\Big)\Big).$$

Particular Solution: We now use the initial value to solve for ϕ. Since $u(x,y) = 0$ when $y = x^2$, we have

$$0 = u(x,y)\big|_{y=x^2} = \ln\Big(x + \phi\Big(\frac{y}{x}\Big)\Big)\Big|_{y=x^2} = \ln(x + \phi(x)) \Rightarrow \phi(x) = 1 - x.$$

Therefore, our particular solution is of the form

$$u(x,y) = \ln\Big(x + 1 - \frac{y}{x}\Big).$$

We can easily verify that these formal computations gives us a solution to the PDE.

Problem 2.3. ($\star\star$) Find a solution to the initial value problem

$$2xyu_x + (x^2+y^2)u_y = 0$$

with $u(x,y) = \exp(x/(x-y))$ on $\{x+y=1\}$.

Solution 2.3. We have the system of equations

$$\frac{dx}{2xy} = \frac{dy}{(x^2+y^2)} = \frac{du}{0}.$$

Characteristic Curves: We start by solving the equation involving the first and second term,

$$\frac{dx}{2xy} = \frac{dy}{(x^2+y^2)} \Rightarrow \frac{dy}{dx} = \frac{1}{2}\cdot\frac{x}{y} + \frac{1}{2}\cdot\frac{y}{x}.$$

This is a Homogenous ODE, which can be solved using the change of variables $w = \frac{y}{x}$. We have $\frac{dy}{dx} = x\frac{dw}{dx} + w$, so under this change of variables we have

$$x\frac{dw}{dx} + w = \frac{1}{2}\cdot w^{-1} + \frac{1}{2}\cdot w \Rightarrow x\frac{dw}{dx} = \frac{1}{2}\cdot w^{-1} - \frac{1}{2}\cdot w = \frac{1-w^2}{2w}.$$

This is a separable equation, so

$$\frac{2w}{1-w^2}dw = \frac{1}{x}dx \Rightarrow -\ln(1-w^2) = \ln x + D \Rightarrow e^{-D} = x(1-w^2) = \frac{x^2-y^2}{x}.$$

If we set $C = e^{-D}$, then $C = \frac{x^2-y^2}{x}$ is our characteristic curve.

General Solution: We now solve the equation involving the first and third term,

$$\frac{dx}{2xy} = \frac{du}{0} \Rightarrow \frac{du}{dx} = 0 \Rightarrow u = \phi(C).$$

Since $C = \frac{x^2-y^2}{x}$, we have our general solution is

$$u(x,y) = \phi\Big(\frac{x^2-y^2}{x}\Big).$$

Particular Solution: We now use the initial value to solve for ϕ. Since $u(x,y) = \exp(x/(x-y))$ when $x+y=1$, we have

$$e^{\frac{x}{x-y}} = u(x,y)\big|_{x+y=1} = \phi\Big(\frac{x^2-y^2}{x}\Big)\Big|_{x+y=1} = \phi\Big(\frac{(x-y)(x+y)}{x}\Big)\Big|_{x+y=1} = \phi\Big(\frac{x-y}{x}\Big).$$

If we set $s = \frac{x-y}{x}$, then the above implies $\phi(s) = e^{\frac{1}{s}}$, so our particular solution is of the form

$$u(x,y) = e^{\frac{x}{x^2-y^2}}.$$

2.2.2 Semilinear Equations in $\mathbb{R}^3$

Problem 2.4. ($\star\star$) Find the general solution to the equation

$$u_x + 3u_y - 2u_z = u.$$

Find the particular solution when $u(0,y,z) = f(y,z)$.

Solution 2.4. We have the system of equations,

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{-2} = \frac{du}{u}.$$

Characteristic Curves Part 1: We start by solving the equation involving the first and second term,

$$\frac{dx}{1} = \frac{dy}{3} \implies C = 3x - y.$$

Characteristic Curves Part 2: We now solve the equation involving the first and third term,

$$\frac{dx}{1} = \frac{dz}{-2} \implies D = 2x + z.$$

General Solution: We now solve the equation involving the first and fourth term,

$$\frac{dx}{1} = \frac{du}{u} \implies x = \log|u| + \psi(C, D) \implies u(x, y, z) = \phi(C, D)e^x = \phi(3x - y, 2x + z)e^x,$$

for some function ϕ of two variables.

Particular Solution: Plugging in our initial conditions, we have

$$u(0, y, z) = \phi(-y, z) = f(y, z).$$

We set $s = -y$ and $t = z$ to conclude that

$$\phi(s, t) = f(-s, t).$$

Therefore, our particular solution is given by

$$u(x, y, z) = \phi(3x - y, 2x + z)e^x = f(y - 3x, 2x + z)e^x.$$

Remark 10. The general form of the solution may be expressed differently depending on which characteristic curves we solve, but the particular solution will be the same. For example, if we solved for the equation involving the second and third term in the second step, we would get

$$C = 3x - y \quad \text{and} \quad D = 3z + 2y.$$

The general solution in this case after solving the equation involving the first and fourth term will result in

$$u(x, y, z) = \phi(C, D)e^x = \phi(3x - y, 3z + 2y)e^x.$$

We can plug in the initial conditions to conclude that

$$u(0, y, z) = \phi(-y, 3z + 2y) = f(y, z).$$

We set $s = -y$ and $t = 3z + 2y$. We write y and z as function of s and t,

$$y = -s \quad \text{and} \quad z = \frac{t - 2y}{3} = \frac{t + 2s}{3} \implies \phi(s, t) = f(y, z) = f\Big(-s, \frac{t + 2s}{3}\Big).$$

Therefore, our particular solution is given by

$$u(x, y, z) = \phi(3x - y, 3z + 2y)e^x = f\left(y - 3x, \frac{3z + 2y + 2(3x - y)}{3}\right)e^x = f(y - 3x, 2x + z)e^x,$$

which is the same as above.

Problem 2.5. ($\star\star\star$) Find the general solution to the equation

$$yu_x + xu_y + u_z = 0.$$

Find the particular solution when $u(x,y,0) = f(x,y)$.

Solution 2.5. We have the system of equations,

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{1} = \frac{du}{0}.$$

Characteristic Curves Part 1: We start by solving the equation involving the first and second term,

$$\frac{dx}{y} = \frac{dy}{x} \implies x^2 = y^2 + C \implies C = x^2 - y^2.$$

Characteristic Curves Part 2: We now solve the equation involving the first and third term using the fact $y = \sqrt{x^2 - C}$,

$$\frac{dz}{1} = \frac{dx}{y} = \frac{dx}{\sqrt{x^2 - C}} \implies z = \log|\sqrt{x^2 - C} + x| + D = \log|y + x| + D \implies D = \frac{(x+y)}{e^z}.$$

General Solution: We now solve the equation involving the third and fourth term,

$$\frac{dz}{1} = \frac{du}{0} \implies u(x,y,z) = \phi(C,D) = \phi\Big(x^2 - y^2, \frac{(x+y)}{e^z}\Big).$$

Particular Solution: Plugging in our initial conditions, we have

$$u(x,y,0) = \phi(x^2 - y^2, x+y) = f(x,y).$$

We set $s = x^2 - y^2$ and $t = x + y$. Our goal is to write x and y as some functions of s and t. We see that

$$s = x^2 - y^2 = (x-y)(x+y) = (x-y)t \implies x - y = \frac{s}{t}.$$

Since $x + y = t$ and $x - y = \frac{s}{t}$, we can add and subtract our answers to conclude

$$x = \frac{1}{2}\Big(t + \frac{s}{t}\Big) \quad \text{and} \quad y = \frac{1}{2}\Big(t - \frac{s}{t}\Big),$$

so

$$\phi(s,t) = f(x,y) = f\Big(\frac{1}{2}\Big(t + \frac{s}{t}\Big), \frac{1}{2}\Big(t - \frac{s}{t}\Big)\Big).$$

Therefore, our particular solution is given by

$$\begin{aligned}
u(x,y,z) &= \phi\Big(x^2 - y^2, \frac{(x+y)}{e^z}\Big) \\
&= f\Big(\frac{1}{2}\Big(\frac{x+y}{e^z} + \frac{x^2 - y^2}{\frac{(x+y)}{e^z}}\Big), \frac{1}{2}\Big(\frac{x+y}{e^z} - \frac{x^2 - y^2}{\frac{(x+y)}{e^z}}\Big)\Big) \\
&= f\Big(\frac{1}{2}\Big(\frac{x+y}{e^z} + \frac{x-y}{e^{-z}}\Big), \frac{1}{2}\Big(\frac{x+y}{e^z} - \frac{x-y}{e^{-z}}\Big)\Big).
\end{aligned}$$

Remark 11. We were a bit sloppy with the constants and domains of our functions above. The constants C and D changed each line and writing $x^2 - y^2 = C$ implicitly in terms of y depends on the value of C. We should check our general solution to ensure that it is a solution to our PDE by differentiating.

2.3 Appendix: Proof of the Method of Characteristics

We now explain why the above method works for certain semilinear PDEs that can be reduced to solvable ODEs. We essentially do a change of variables to reduce the PDE into an ODE. Suppose we have a PDE of the form

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u). \tag{$*$}$$

Step 1: We want to solve the equation

$$\frac{dx}{a(x,y)} = \frac{dy}{b(x,y)} \Rightarrow \frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}.$$

Suppose we can find a family of solutions $f(x,y) = C$, called the characteristic curves. Since this is a solution to the ODE, by implicitly differentiating, we must have

$$0 = \frac{d}{dx}f(x,y) = f_x(x,y) + f_y(x,y)\frac{dy}{dx} = f_x(x,y) + f_y(x,y)\frac{b(x,y)}{a(x,y)}$$

and therefore, our the family of solutions must satisfy the condition

$$0 = a(x,y)f_x(x,y) + b(x,y)f_y(x,y). \tag{14}$$

We will see that using this $f(x,y)$, we can reduce our PDE into either an ODE with respect to x (Choice 1) or an ODE with respect to y (Choice 2).

Step 2 (Choice 1): We now explain why it suffices to solve the equations

$$\frac{dx}{a(x,y)} = \frac{du}{c(x,y,u)} \Rightarrow \frac{du}{dx} = \frac{c(x,y,u)}{a(x,y)}.$$

To reduce ($*$) to this ODE, we do a change of variables

$$\xi(x,y) = x, \quad \eta(x,y) = f(x,y)$$

where $f(x,y) = C$ is the function we found in Step 1. Notice that

$$u_x = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = u_\xi + u_\eta f_x(x,y)$$

and

$$u_y = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial y} = u_\eta f_y(x,y)$$

therefore we have

$$\begin{aligned} a(x,y)u_x + b(x,y)u_y &= a(x,y)(u_\xi f_x(x,y)u_\eta) + b(x,y)f_y(x,y)u_\eta \\ &= a(x,y)u_\xi + (a(x,y)f_x(x,y) + b(x,y)f_y(x,y))u_\eta \\ &= a(x,y)u_\xi(\xi,\eta) \end{aligned}$$

since $f(x,y)$ satisfies (14). Since our PDE satisfies ($*$), we have shown that

$$u_\xi(\xi,\eta) = \frac{c(x(\xi,\eta),y(\xi,\eta),u)}{a(x(\xi,\eta),y(\xi,\eta))}$$

We can write this back in our original coordinates $\xi = x$, $\eta = f(x,y) = C$. If the characteristic curve $f(x,y) = C$ can be solved implictly for $y = y(x,C)$, then we can eliminate the y variable, which gives us

$$u_x(x,C) = \frac{c(x,y(x,C),u)}{a(x,y(x,C))}.$$

This is precisely the ODE we are solving if we choose the first ODE in Step 3. This also explains why the integration constant is of the form $F(C)$ instead of just a constant.

Step 2 (Choice 2): It might be easier to instead solve the system

$$\frac{dy}{b(x,y)} = \frac{du}{c(x,y,u)} \Rightarrow \frac{du}{dy} = \frac{c(x,y,u)}{b(x,y)}.$$

To reduce $(*)$ to this ODE, we instead use the change of variables,

$$\xi(x,y) = y, \quad \eta(x,y) = f(x,y),$$

where $f(x,y) = C$ is the function we found in Step 1. Notice that

$$u_x = \frac{\partial u}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = u_\eta f_x(x,y)$$

and

$$u_y = \frac{\partial u}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = u_\xi + u_\eta f_y(x,y)$$

so we have

$$\begin{aligned} a(x,y)u_x + b(x,y)u_y &= a(x,y)(f_x(x,y)u_\eta) + b(x,y)(u_\xi + f_y(x,y)u_\eta) \\ &= b(x,y)u_\xi(\xi,\eta) + (a(x,y)f_x(x,y) + b(x,y)f_y(x,y))u_\eta \\ &= b(x,y)u_\xi(\xi,\eta) \end{aligned}$$

since $f(x,y)$ satisfies (14). Since our PDE satisfies $(*)$, we have shown that

$$u_\xi(\xi,\eta) = \frac{c(x(\xi,\eta), y(\xi,\eta), u)}{a(x(\xi,\eta), y(\xi,\eta))}.$$

We can write this back in our original coordinates $\xi = y$, $\eta = f(x,y) = C$. If the characteristic curve $f(x,y) = C$ can be solved implictly for $x = x(y,C)$, then we can eliminate the x variable, which gives us

$$u_y(C,y) = \frac{c(x(y,C), y, u)}{a(x(y,C), y)}.$$

This is precisely the ODE we are solving if we choose the second ODE in Step 3. This also explains why the integration constant is of the form $F(C)$ instead of just a constant.

Remark 12. Notice that the above procedure works even if one of the coefficients vanish (i.e. $a \equiv 0$, $b \equiv 0$ or $c \equiv 0$). We just lose some choice as to the equations we solve, since if $a \equiv 0$, then we must use Choice 2 because Choice 1 does not make sense. The derivation assumed that the coefficients were sufficiently nice such that we can find solutions to the ODEs at least locally. We should verify that our formal solutions are true solutions by checking our solutions after deriving them.