1 Laplace's Equation on Circular Domains

Laplace's equation on rotationally symmetric domains can be solved using a change of variables to polar coordinates. The two dimensional Laplace operator in its Cartesian and polar forms are

$$\Delta u(x,y) = u_{xx} + u_{yy} \quad \text{and} \quad \Delta u(r,\theta) = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

We are interested in finding bounded solutions to Laplace's equation, so we often have that implicit assumption. The "radial" problem will be an Euler ODE which has the following solution.

Euler Equations: An ODE of the form

$$ax^2y'' + bxy' + cy = 0$$

are called Euler ODEs. The ODE is solved by finding the roots r_1 and r_2 of the characteristic polynomial

$$C(x) = ax(x-1) + bx + c = 0$$

and the general form of the solution is given by

$$y(x) = \begin{cases} C_1x^{r_1} + C_2x^{r_2} & r_1, r_2 \in \mathbb{R}, r_1 \neq r_2 \\ C_1x^r + C_2\log(x)x^r & r_1 = r_2 = r \in \mathbb{R} \\ C_1x^\alpha\cos(\beta\log x) + C_2x^\alpha\sin(\beta\log x) & r_1 = \alpha + i\beta, r_2 = \alpha - i\beta, \beta \neq 0. \end{cases}$$

1.1 Rotational Invariance

The reason why the Laplacian has a simple form in polar coordinates is because it is invariant under rotations. Given α, consider the counterclockwise rotation by α,

$$\begin{bmatrix}\tilde{x}\\ \tilde{y}\end{bmatrix} = \begin{bmatrix}\cos(\alpha) & -\sin(\alpha)\\ \sin(\alpha) & \cos(\alpha)\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} \implies \begin{matrix}\tilde{x} = \cos(\alpha)x - \sin(\alpha)y\\ \tilde{y} = \sin(\alpha)x + \cos(\alpha)y\end{matrix}.$$

By the chain rule,

$$\begin{aligned}\partial_x &= \cos(\alpha)\partial_{\tilde{x}} + \sin(\alpha)\partial_{\tilde{y}}\\ \partial_y &= -\sin(\alpha)\partial_{\tilde{x}} + \cos(\alpha)\partial_{\tilde{y}}\end{aligned} \implies \begin{aligned}\partial_x^2 &= \cos^2(\alpha)\partial_{\tilde{x}}^2 + 2\sin(\alpha)\cos(\alpha)\partial_{\tilde{x}}\partial_{\tilde{y}} + \sin^2(\alpha)\partial_{\tilde{y}}^2\\ \partial_y^2 &= \sin^2(\alpha)\partial_{\tilde{x}}^2 - 2\sin(\alpha)\cos(\alpha)\partial_{\tilde{x}}\partial_{\tilde{y}} + \cos^2(\alpha)\partial_{\tilde{y}}^2\end{aligned}$$

so

$$u_{xx} + u_{yy} = (\cos^2\alpha + \sin^2\alpha)(u_{\tilde{x}\tilde{x}} + u_{\tilde{y}\tilde{y}}) = u_{\tilde{x}\tilde{x}} + u_{\tilde{y}\tilde{y}}.$$

This can also be seen in its polar form. If we define $\tilde{\theta} = \theta + \alpha$, then

$$\partial_\theta = \partial_{\tilde{\theta}} \qquad \partial_\theta^2 = \partial_{\tilde{\theta}}^2$$

so

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\tilde{\theta}\tilde{\theta}}.$$

Remark 1. This property generalizes nicely to n dimensions as well. Given a rotation matrix $\boldsymbol{R}$, consider the rotation $\tilde{x} = \boldsymbol{R}\vec{x}$. If we define $\nabla_{\vec{x}} = (\partial_{x_1}, \dots, \partial_{x_n})^\mathsf{T}$ and $\nabla_{\tilde{x}} = (\partial_{\tilde{x}_1}, \dots, \partial_{\tilde{x}_n})^\mathsf{T}$, then the chain rule implies

$$\nabla_{\vec{x}} = \boldsymbol{R}^\mathsf{T}\nabla_{\tilde{x}}$$

so it follows that

$$\Delta_{\vec{x}} = \nabla_{\vec{x}}^\mathsf{T}\nabla_{\vec{x}} = (\boldsymbol{R}^\mathsf{T}\nabla_{\tilde{x}})^\mathsf{T}(\boldsymbol{R}^\mathsf{T}\nabla_{\tilde{x}})_{\vec{x}} = \nabla_{\tilde{x}}^\mathsf{T}\boldsymbol{R}\boldsymbol{R}^\mathsf{T}\nabla_{\tilde{x}} = \nabla_{\tilde{x}}^\mathsf{T}\nabla_{\tilde{x}} = \Delta_{\tilde{x}},$$

since rotation matrices are orthogonal $\boldsymbol{R}\boldsymbol{R}^\mathsf{T} = \boldsymbol{I}$. It follows that $\Delta_{\vec{x}}u = \Delta_{\tilde{x}}u$, so the Laplacian in $\mathbb{R}^n$ is also rotationally invariant.

1.2 Example Problems

Problem 1.1. ($\star$) (Interior of a Disk) Solve

$$\begin{cases} \Delta u = 0 & \text{for } r < a, \quad -\pi \le \theta \le \pi, \\ u|_{r=a} = 1 + 2\sin(\theta), \end{cases}$$

Solution 1.1. After converting to polar coordinates, our PDE can be written as the following problem on the circle

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 & 0 < r < a, \quad -\pi \le \theta \le \pi \\ u(r,-\pi) = u(r,\pi) & 0 < r < a \\ u_\theta(r,-\pi) = u_\theta(r,\pi) & 0 < r < a \\ u(a,\theta) = 1 + 2\sin(\theta) & -\pi \le \theta \le \pi \\ \lim_{r\to 0} u(r,\theta) < \infty & -\pi \le \theta \le \pi \end{cases}$$

The condition that $\lim_{r\to 0} u(r,\theta) < \infty$ is an implicit assumption of this problem.

Step 1 — Separation of Variables: The PDE has periodic homogeneous angular boundary conditions, so we look for a solution of the form $u(r,\theta) = R(r)\Theta(\theta)$. For such a solution, the PDE implies

$$\Delta u = R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \implies -\frac{r^2R'' + rR'}{R} = \frac{\Theta''}{\Theta} = -\lambda.$$

This results in the ODEs

$$r^2R''(r) + rR'(r) - \lambda R(r) = 0 \text{ and } \Theta''(\theta) + \lambda\Theta(\theta) = 0$$

with angular boundary conditions

$$R(r)\Theta(-\pi) - R(r)\Theta(\pi) = 0, \quad R(r)\Theta'(-\pi) - R(r)\Theta'(\pi) = 0$$

and radial boundary conditions

$$R(a)\Theta(\theta) = 1 + 2\sin(\theta) \qquad \lim_{r\to 0} R(r)\Theta(\theta) < \infty.$$

For non-trivial solutions to the angle problem, we require $R(r) \not\equiv 0$, $\Theta(-\pi) = \Theta(\pi)$, $\Theta'(-\pi) = \Theta'(\pi)$.

Step 2 — Eigenvalue Problem: We now solve the periodic angular eigenvalue problem

$$\begin{cases} \Theta'' + \lambda\Theta = 0 \\ \Theta(\pi) - \Theta(-\pi) = \Theta'(\pi) - \Theta'(-\pi) = 0. \end{cases}$$

The eigenvalues and corresponding eigenfunctions are given by the full Fourier series

$$\lambda_0 = 0,\ \Theta_0(x) = 1, \quad \lambda_n = n^2,\ \Theta_n(\theta) = \cos(n\theta),\ \Phi_n(\theta) = \sin(n\theta), \quad n = 1,2,\dots.$$

Step 3 — Radial Problem: We now solve the radial problem for each eigenvalue. The ODE

$$r^2R'' + rR' - \lambda R = 0$$

is an Euler ODE with solutions

$$R_0(r) = C_0 \log r + D_0, \qquad R_n(r) = C_n r^{-n} + D_n r^n, \qquad n = 1,2,\dots$$

Since the solution should be regular at 0 ($\lim_{r\to 0} R(r) < \infty$), we need $C_n = 0$ for all $n \geq 0$, so our solution is of the form

$$R_0(r) = D_0, \qquad R_n(r) = D_n r^n, \qquad n = 1, 2, \dots$$

for some arbitrary coefficients D_0 and D_n.

Step 4 — General Solution: Using the principle of superposition, and summing all the eigenfunctions gives us the general solution

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right)$$

where A_0, A_n and B_n are yet to be determined coefficients.

Step 5 — Particular Solution: To find constants A_0, A_n and B_n we need to use the boundary condition. Using the boundary condition we get

$$1 + 2\sin(\theta) = u(a,\theta) = A_0 + \sum_{n=1}^{\infty} a^n \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right).$$

Instead of solving for the Fourier series like usual, we can just equate coefficients to see that

$$A_0 = 1, \qquad a^1 B_1 = 2 \implies B_1 = \frac{2}{a},$$

and the rest of the coefficients are 0.

Step 6 — Final Answer: To summarize, the solution to the PDE is given by

$$u(r,\theta) = 1 + \frac{2}{a} r \sin(\theta).$$

Problem 1.2. ($\star\star$) (Wedge of a Disk) Solve

$$\begin{cases} \Delta u = 0 & \text{for } r < a,\ 0 < \theta < \beta, \\ u|_{r=a} = g(\theta), \\ u|_{\theta=0} = 0,\ u|_{\theta=\beta} = \beta \end{cases}$$

Solution 1.2. After converting to polar coordinates, our PDE can be written as the following problem on the wedge

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 & 0 < r < a, \quad 0 < \theta < \beta \\ u(r,0) = 0 & 0 < r < a \\ u(r,\beta) = \beta & 0 < r < a \\ u(a,\theta) = g(\theta) & 0 < \theta < \beta \end{cases}$$

Step 1 — Change of Variables: Before doing separation of variables, we begin by using a change of variables to reduce our problem to the case with homogeneous angular boundary conditions. We set

$$u(r,\theta) = v(r,\theta) + w(r,\theta)$$

where $w(r,\theta)$ is chosen to satisfy the inhomogeneous boundary conditions. Like usual, we can take $w(r,\theta)$ to be a polynomial of the form

$$w(r,\theta) = (A\theta^2 + B\theta + C) \cdot \beta$$

for some constants A, B, C. Substituting $w(r,\theta)$ in the boundary conditions gives

$$\begin{aligned} C &= 0 = w(r,0) \\ (A\beta^2 + B\beta + C)\beta &= \beta = w(r,\beta). \end{aligned}$$

By inspection it is clear that $A = 0$, $B = 1/\beta$, and $C = 0$ zero works. Therefore,

$$w(r,\theta) = \theta.$$

Step 2 — Separation of Variables: Since $v(r,\theta) = u(r,\theta) - w(r,\theta)$, our choice of $w(r,\theta)$ implies

$$\begin{cases} v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = 0 & 0 < r < a, \quad 0 < \theta < \beta \\ v(r,0) = 0 & 0 < r < a \\ v(r,\beta) = 0 & 0 < r < a \\ v(a,\theta) = g(\theta) - \theta & 0 < \theta < \beta. \end{cases} \tag{$*$}$$

This now has homogeneous angular boundary conditions, so we can use separation of variables and look for a solution of the form $v(r,\theta) = R(r)\Theta(\theta)$. For such a solution, the PDE implies

$$\Delta v = R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \implies -\frac{r^2R'' + rR'}{R} = \frac{\Theta''}{\Theta} = -\lambda.$$

This results in the ODEs

$$r^2R''(r) + rR'(r) - \lambda R(r) = 0 \text{ and } \Theta''(\theta) + \lambda\Theta(\theta) = 0$$

with angular boundary conditions

$$R(r)\Theta(0) = R(r)\Theta(\beta) = 0.$$

and radial boundary conditions (and regularity condition)

$$R(a)\Theta(\theta) = \theta, \quad \lim_{r\to 0} R(r)\Theta(\theta) < \infty$$

For non-trivial solutions to the angle problem, we require $R(r) \not\equiv 0$, $\Theta(0) = \Theta(\beta) = 0$.

Step 3 — Eigenvalue Problem: We now solve the angular eigenvalue problem

$$\begin{cases} \Theta''(\theta) + \lambda\Theta(\theta) = 0 & 0 < \theta < \beta \\ \Theta(0) = \Theta(\beta) = 0. \end{cases}$$

This is a standard problem, and the eigenvalues and corresponding eigenfunctions are

$$\lambda_n = \Big(\frac{n\pi}{\beta}\Big)^2, \ \Theta_n(\theta) = \sin\Big(\frac{n\pi}{\beta}\theta\Big), \quad n = 1, 2, \dots.$$

Step 4 — Radial Problem: For each eigenvalue, we solve the radial problem

$$r^2R''(r) + rR'(r) - \Big(\frac{n\pi}{\beta}\Big)^2 R(r) = 0.$$

This is an Euler ODE with characteristic equation $C(r) = r(r-1) + r - (\frac{n\pi}{\beta})^2$ and roots $r = \pm\frac{n\pi}{\beta}$, which has general solution of the form

$$R_n(r) = A_n r^{\frac{n\pi}{\beta}} + B_n r^{-\frac{n\pi}{\beta}}$$

for some yet to be determined coefficients A_n and B_n. Since the solution should be regular at 0 ($\lim_{r\to 0} R(r) < \infty$), we need $B_n = 0$, so our solution is of the form

$$R_n(r) = A_n r^{\frac{n\pi}{\beta}}, \qquad n = 1, 2, \dots$$

for some yet to be determined coefficient A_n. Using the principle of superposition, and taking a linear combination of the eigenfunctions gives the general solution

$$v(r,\theta) = \sum_{n=1}^{\infty} A_n r^{\frac{n\pi}{\beta}} \sin\Big(\frac{n\pi}{\beta}\theta\Big).$$

Step 5 — Particular Solution: We now use the radial boundary condition to find A_n. Plugging the general solution into the boundary conditions, $R(a)\Theta(\theta) = g(\theta) - \theta$ implies

$$\sum_{n=1}^{\infty} A_n a^{\frac{n\pi}{\beta}} \sin\Big(\frac{n\pi}{\beta}\theta\Big) = g(\theta) - \theta.$$

By the Fourier sine series, we have

$$a^{\frac{n\pi}{\beta}} A_n = \frac{2}{\beta}\int_0^{\beta} (g(\theta) - \theta) \sin\Big(\frac{n\pi}{\beta}\theta\Big)\, d\theta \implies A_n = a^{-\frac{n\pi}{\beta}} \frac{2}{\beta}\int_0^{\beta} (g(\theta) - \theta) \sin\Big(\frac{n\pi}{\beta}\theta\Big)\, d\theta.$$

Step 6 — Final Answer: To summarize, since $u(r,\theta) = v(r,\theta) + w(r,\theta)$ we have

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{\frac{n\pi}{\beta}} \sin\Big(\frac{n\pi}{\beta}\theta\Big) + \theta.$$

where the coefficients A_n are given by

$$A_n = a^{-\frac{n\pi}{\beta}} \frac{2}{\beta}\int_0^{\beta} (g(\theta) - \theta) \sin\Big(\frac{n\pi}{\beta}\theta\Big)\, d\theta.$$

Problem 1.3. ($\star\star$) (Wedge of an Annulus) Solve

$$\begin{cases} \Delta u = 0 & \text{for } a < r < b,\ \alpha < \theta < \beta, \\ u|_{r=a} = g(\theta),\ u|_{r=b} = h(\theta), \\ u|_{\theta=\alpha} = u|_{\theta=\beta} = 0. \end{cases}$$

Solution 1.3. After converting to polar coordinates, our PDE can be written as the following problem on the wedge of an annuli

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 & 0 < a < r < b, \quad \alpha < \theta < \beta \\ u(r,\alpha) = 0 & 0 < a < r < b \\ u(r,\beta) = 0 & 0 < a < r < b \\ u(a,\theta) = g(\theta) & \alpha < \theta < \beta \\ u(b,\theta) = h(\theta) & \alpha < \theta < \beta \end{cases}$$

Step 1 — Change of Variables: Before doing separation of variables, we begin by using a change of variables to reduce our problem to the case with the standard Dirichlet angular boundary conditions. We use rotation invariance, and set

$$v(r,\theta) = u(r,\theta + \alpha).$$

By rotational invariance, it is easy to see that $v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ and the domain of $v(r,\theta)$ is the centered wedge of the annuli $\{0 < \theta < \beta - \alpha, a < r < b\}$.

Step 2 — Separation of Variables: Since $v(r,\theta) = u(r,\theta+\alpha)$, our PDE can be expressed as

$$\begin{cases} v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = 0 & 0 < a < r < b, \quad 0 < \theta < \beta - \alpha \\ v(r,0) = 0 & 0 < a < r < b \\ v(r,\beta-\alpha) = 0 & 0 < a < r < b \\ v(a,\theta) = g(\theta+\alpha) & 0 < \theta < \beta - \alpha \\ v(b,\theta) = h(\theta+\alpha) & 0 < \theta < \beta - \alpha. \end{cases}$$

This PDE now has symmetric homogeneous angular boundary conditions, so we look for a solution of the form $v(r,\theta) = R(r)\Theta(\theta)$. For such a solution, the PDE implies

$$\Delta v = R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \implies -\frac{r^2R'' + rR'}{R} = \frac{\Theta''}{\Theta} = -\lambda.$$

This results in the ODEs

$$r^2R''(r) + rR'(r) - \lambda R(r) = 0 \text{ and } \Theta''(\theta) + \lambda\Theta(\theta) = 0$$

with angular boundary conditions

$$R(r)\Theta(0) = R(r)\Theta(\beta-\alpha) = 0,$$

and radial boundary conditions

$$R(a)\Theta(\theta) = g(\theta+\alpha), \quad R(b)\Theta(\theta) = h(\theta+\alpha).$$

For non-trivial solutions to the angle problem, we require $R(r) \not\equiv 0$, $\Theta(0) = \Theta(\beta-\alpha) = 0$.

Step 3 — Eigenvalue Problem: We now solve the angular eigenvalue problem

$$\begin{cases} \Theta''(\theta) + \lambda\Theta(\theta) = 0 & 0 < \theta < \beta - \alpha \\ \Theta(0) = \Theta(\beta-\alpha) = 0. \end{cases}$$

This is a standard problem, and the eigenvalues and corresponding eigenfunctions are

$$\lambda_n = \Big(\frac{n\pi}{\beta-\alpha}\Big)^2, \ \Theta_n(\theta) = \sin\Big(\frac{n\pi}{\beta-\alpha}\theta\Big), \quad n = 1,2,\dots.$$

Step 4 — Radial Problem: For each eigenvalue, we solve the radial problem

$$r^2R''(r) + rR'(r) - \Big(\frac{n\pi}{\beta-\alpha}\Big)^2 R(r) = 0.$$

This is an Euler ODE with characteristic equation $C(r) = r(r-1) + r - (\frac{n\pi}{\beta-\alpha})^2$ and roots $r = \pm\frac{n\pi}{\beta-\alpha}$, which has general solution of the form

$$R_n(r) = A_n r^{\frac{n\pi}{\beta-\alpha}} + B_n r^{-\frac{n\pi}{\beta-\alpha}}$$

for some yet to be determined coefficients A_n and B_n. Using the principle of superposition, and taking a linear combination of the eigenfunctions gives the general solution

$$v(r,\theta) = \sum_{n=1}^{\infty} \big(A_n r^{\frac{n\pi}{\beta-\alpha}} + B_n r^{-\frac{n\pi}{\beta-\alpha}}\big)\sin\Big(\frac{n\pi}{\beta-\alpha}\theta\Big).$$

Step 5 — Particular Solution: We now use the radial boundary condition to find A_n. Plugging the general solution into the radial boundary conditions implies,

$$\sum_{n=1}^{\infty} \big(A_n a^{\frac{n\pi}{\beta-\alpha}} + B_n a^{-\frac{n\pi}{\beta-\alpha}}\big) \sin\Big(\frac{n\pi}{\beta-\alpha}\theta\Big) = g(\theta+\alpha)$$

and

$$\sum_{n=1}^{\infty} \big(A_n b^{\frac{n\pi}{\beta-\alpha}} + B_n b^{-\frac{n\pi}{\beta-\alpha}}\big) \sin\Big(\frac{n\pi}{\beta-\alpha}\theta\Big) = h(\theta+\alpha).$$

By the Fourier sine series, we have

$$A_n a^{\frac{n\pi}{\beta-\alpha}} + B_n a^{-\frac{n\pi}{\beta-\alpha}} = \frac{2}{\beta-\alpha}\int_0^{\beta-\alpha} g(\theta+\alpha)\sin\Big(\frac{n\pi}{\beta-\alpha}\theta\Big)\,d\theta = \frac{2}{\beta-\alpha}\int_\alpha^\beta g(\theta)\sin\Big(\frac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta,$$

and

$$A_n b^{\frac{n\pi}{\beta-\alpha}} + B_n b^{-\frac{n\pi}{\beta-\alpha}} = \frac{2}{\beta-\alpha}\int_0^{\beta-\alpha} h(\theta+\alpha)\sin\Big(\frac{n\pi}{\beta-\alpha}\theta\Big)\,d\theta = \frac{2}{\beta-\alpha}\int_\alpha^\beta h(\theta)\sin\Big(\frac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta.$$

This system can be written as a 2×2 matrix with linearly independent columns (since $a \neq b$), so we may solve for A_n and B_n. To solve this system, we can use the formula for the inverse of a 2×2 matrix. If we define $I_n = \frac{2}{\beta-\alpha}\int_\alpha^\beta g(\theta)\sin\Big(\frac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta$ and $J_n = \frac{2}{\beta-\alpha}\int_\alpha^\beta h(\theta)\sin\Big(\frac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta$, this gives

$$\begin{bmatrix} a^{\frac{n\pi}{\beta-\alpha}} & a^{-\frac{n\pi}{\beta-\alpha}} \\ b^{\frac{n\pi}{\beta-\alpha}} & b^{-\frac{n\pi}{\beta-\alpha}} \end{bmatrix}\begin{bmatrix} A_n \\ B_n\end{bmatrix} = \begin{bmatrix} I_n \\ J_n\end{bmatrix} \implies \begin{bmatrix} A_n \\ B_n\end{bmatrix} = \frac{1}{a^{\frac{n\pi}{\beta-\alpha}} b^{-\frac{n\pi}{\beta-\alpha}} - a^{-\frac{n\pi}{\beta-\alpha}} b^{\frac{n\pi}{\beta-\alpha}}}\begin{bmatrix} b^{-\frac{n\pi}{\beta-\alpha}} & -a^{-\frac{n\pi}{\beta-\alpha}} \\ -b^{\frac{n\pi}{\beta-\alpha}} & a^{\frac{n\pi}{\beta-\alpha}} \end{bmatrix}\begin{bmatrix} I_n \\ J_n\end{bmatrix}.$$

That is, for $C_n = a^{\frac{n\pi}{\beta-\alpha}} b^{-\frac{n\pi}{\beta-\alpha}} - a^{-\frac{n\pi}{\beta-\alpha}} b^{\frac{n\pi}{\beta-\alpha}}$ we have

$$A_n = \frac{1}{C_n}\cdot\bigg(\frac{2b^{-\frac{n\pi}{\beta-\alpha}}}{\beta-\alpha}\int_\alpha^\beta g(\theta)\sin\Big(\frac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta - \frac{2a^{-\frac{n\pi}{\beta-\alpha}}}{\beta-\alpha}\int_\alpha^\beta h(\theta)\sin\Big(\frac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta\bigg)$$

and

$$B_n = \frac{1}{C_n}\cdot\bigg(-\frac{2b^{\frac{n\pi}{\beta-\alpha}}}{\beta-\alpha}\int_\alpha^\beta g(\theta)\sin\Big(\frac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta + \frac{2a^{\frac{n\pi}{\beta-\alpha}}}{\beta-\alpha}\int_\alpha^\beta h(\theta)\sin\Big(\frac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta\bigg).$$

Step 6 — Final Answer: To summarize, since $v(r,\theta) = u(r,\theta+\alpha)$ we have $u(r,\theta) = v(r,\theta-\alpha)$, we have

$$u(r,\theta) = \sum_{n=1}^{\infty}\big(A_n r^{\frac{n\pi}{\beta-\alpha}} + B_n r^{-\frac{n\pi}{\beta-\alpha}}\big)\sin\Big(\frac{n\pi}{\beta-\alpha}(\theta-\alpha)\Big),$$

where for $C_n = a^{\frac{n\pi}{\beta-\alpha}} b^{-\frac{n\pi}{\beta-\alpha}} - a^{-\frac{n\pi}{\beta-\alpha}} b^{\frac{n\pi}{\beta-\alpha}}$ the coefficients A_n and B_n are given by

$$A_n = \frac{1}{C_n}\cdot\bigg(\frac{2b^{-\frac{n\pi}{\beta-\alpha}}}{\beta-\alpha}\int_\alpha^\beta g(\theta)\sin\Big(\frac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta - \frac{2a^{-\frac{n\pi}{\beta-\alpha}}}{\beta-\alpha}\int_\alpha^\beta h(\theta)\sin\Big(\frac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta\bigg)$$

and

$$B_n = \frac{1}{C_n}\cdot\bigg(-\frac{2b^{\frac{n\pi}{\beta-\alpha}}}{\beta-\alpha}\int_\alpha^\beta g(\theta)\sin\Big(\frac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta + \frac{2a^{\frac{n\pi}{\beta-\alpha}}}{\beta-\alpha}\int_\alpha^\beta h(\theta)\sin\Big(\frac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big)\,d\theta\bigg).$$

Remark 2. If we solve the following eigenvalue problem

$$\begin{cases} \Theta''(\theta) + \lambda\Theta(\theta) = 0 & \alpha < \theta < \beta, \\ \Theta(\alpha) = \Theta(\beta) = 0, \end{cases}$$

we will get the eigenvalue and eigenfunctions

$$\lambda_n = \Big(\frac{n\pi}{\beta-\alpha}\Big)^2 \quad \Theta_n(\theta) = \sin\Big(\frac{n\pi(\theta-\alpha)}{\beta-\alpha}\Big) \quad n = 1, 2, \dots.$$