REPRESENTATION THEORY
ASSIGNMENT 2
DUE FRIDAY FRIDAY FEBRUARY 18

(1) Show that there is a isomorphism of Lie groups $\mathbb{R}^\times \to \mathbb{R} \times \mathbb{Z}/2$ (where $\mathbb{R}^\times$ is a group under multiplication and $\mathbb{R}$ is a group under addition). Use this fact to show that $\mathbb{R}^\times$ has two different structure of a real algebraic group. Show that these two different structures are non-isomorphic by showing that their complexifications are different.

(2) (a) Show that every algebraic representation of $GL_n(\mathbb{R})$ on (finite-dimensional) complex vector space is complete reducible (i.e. is the direct sum of irreducible subrepresentations).

(b) Give an example of a non-algebraic representation of $GL_n(\mathbb{R})$ on a (finite-dimensional) complex vector space which is not complete reducible.

(3) Consider $G = SO_{2n}(\mathbb{C})$. Find the roots and coroots of G as well the $\psi_\alpha : SL_2(\mathbb{C}) \to SO_{2n}(\mathbb{C})$.

Here is a suggestion to help you get started. Recall that $SO_{2n}(\mathbb{C})$ is the automorphisms of $\mathbb{C}^{2n}$ which preserve a non-degenerate symmetric bilinear form $\langle , \rangle$. Choose a basis

$$v_1, \dots, v_n, v_{-1}, \dots, v_n$$

for $\mathbb{C}^{2n}$ such that

$$\langle v_i, v_j \rangle = \begin{cases} 1, & \text{if } i = j \pm n \\ 0, & \text{otherwise} \end{cases}$$

Then the maximal torus is given by those elements of $SO_{2n}(\mathbb{C})$ which are diagonal with respect to this basis.

(4) Consider the group $GO_{2n}(\mathbb{C})$ which is called the orthogonal similitude group. It consists of those automorphisms of $\mathbb{C}^{2n}$ which preserve the bilinear form up to a scalar. In other words for each $g \in G$, there exists a scalar $a \in \mathbb{C}^\times$ such that $\langle gv, gw \rangle = a\langle v, w \rangle$ for all $v, w \in \mathbb{C}^{2n}$. Find the root datum of $GO_{2n}(\mathbb{C})$ and compare with $SO_{2n}(\mathbb{C})$.

(5) Show that $\Lambda^2\mathbb{C}^4$ carries a natural non-degenerate symmetric bilinear form. Use this fact to define a 2-to-1 cover $SL_4(\mathbb{C}) \to SO_6(\mathbb{C})$. What does this map look like on the level of root data?