MAT 247, Winter 2014
Assignment 7
Due March 11

1. Let $\{V_i\}_{i\in I}$ be a collection (possibly infinite) of vector spaces. There are two ways to take the "direct sum" of all these vector spaces. First we have the direct sum

$$\bigoplus_{i\in I} V_i := \{(v_i)_{i\in I} : v_i \in V_i \text{ and } v_i \text{ is non-zero for only finitely-many } i\}$$

and we have the direct product

$$\prod_{i\in I} V_i := \{(v_i)_{i\in I} : v_i \in V_i\}$$

In each case, they are vector spaces, with addition and scalar multiplication defined in the obvious way.

In each case we have inclusion map $\phi_i : V_i \to \bigoplus_{i\in I} V_i$ and $\phi_i : V_i \to \prod_{i\in I} V_i$ and projection maps $\psi_i : \bigoplus_{i\in I} V_i \to V_i$ and $\psi_i : \prod_{i\in I} V_i \to V_i$.

For each of the two following statements, fill in the blank with either the direct sum or the direct product and then prove the statement.

   (a) Let X be a vector space and let $T_i : V_i \to X$ be linear maps for all $i \in I$. There exists a unique linear map $T :$ _____ $\to X$ such that $T_i = T \circ \phi_i$ for all i.

   (b) Let X be a vector space and let $U_i : X \to V_i$ be linear maps for all $i \in I$. There exists a unique linear map $U : X \to$ _____ such that $U_i = \psi_i \circ U$ for all i.

2. Let I be any set and let

$$\mathbb{F}[I] = \{(a_i)_{i \in I} : a_i \in \mathbb{F} \text{ is non-zero for only finitely-many } i\}$$

Let $e_i \in \mathbb{F}[I]$ be the "tuple" which is 1 in the ith slot and 0 elsewhere.

Let X be a vector space and for each $i \in I$, let $x_i \in X$. Prove that there exists a unique linear map $T : \mathbb{F}[I] \to X$ such that $T(e_i) = x_i$ for all $i \in I$.

3. Given an example of an element of $\mathbb{F}^2 \otimes \mathbb{F}^2$ which cannot be written as $v \otimes w$.

4. Let V and W be vector spaces. If $\alpha \in V^*$ and $w \in W$, define $T_{\alpha,w} : V \to W$ by $T_{\alpha,w}(v) = \alpha(v)w$.

   (a) Prove that for any α, w, $T_{\alpha,w}$ is a linear map.

   (b) Define a linear map $\psi : V^* \otimes W \to L(V, W)$ by $\psi(\alpha \otimes w) = T_{\alpha,w}$. Prove that ψ is well-defined and that it is an isomorphism of vector spaces when V, W are finite-dimensional.

   (c) Let $T \in L(V, W)$. Prove that $T = \psi(\alpha \otimes w)$ for some $\alpha \in V^*, w \in W$ if and only if $rank(T) \leq 1$.

5. (a) Let A and B be upper-triangular square matrices. Prove that $A \otimes B$ is also upper triangular.

   (b) Let $T : V \to V$ and $U : W \to W$ be linear operators. We have the linear operator $T \otimes U : V \otimes W \to V \otimes W$. If λ is an eigenvalue of T and μ is an eigenvalue of U, prove that $\lambda\mu$ is an eigenvalue of $T \otimes U$.

   (c) Assume $\mathbb{F} = \mathbb{C}$. Use (a) to prove that every eigenvalue of $T \otimes U$ can be written as $\lambda\mu$ where λ is an eigenvalue of T and μ is an eigenvalue of U.