MAT 247, Winter 2014
Assignment 6
Due Feb 25

1. Let $T : V \to V$ be a linear operator and let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues of T.

   (a) Assume that T is diagonalizable. Prove a subspace $W \subset V$ is T-invariant if and only if

   $$W = \bigoplus_{i=1}^{k} W \cap E_{\lambda_i}$$

   (b) Suppose that T is not diagonalizable, but that $\mathbb{F} = \mathbb{C}$. In class we proved that if W is T-invariant, then

   $$W = \bigoplus_{i=1}^{k} W \cap K_{\lambda_i}$$

   Give a example to show that the converse is false.

   (c) Suppose that T is diagonalizable and that all eigenspaces are 1-dimensional. Find the number of T-invariant subspaces of V.

   (d) Suppose that T is the linear operator defined by a single $n\times n$ Jordan block. Find the number of T-invariant subspaces of V.

2. Let V be a vector space (not necessarily finite-dimensional). Define a map $\phi : V \to (V^*)^*$ by $\phi(v)(\alpha) = \alpha(v)$.

   (a) Prove that ϕ is a linear map.

   (b) Prove that ϕ is injective.

(c) Prove that ϕ is an isomorphism when V is finite-dimensional.

(d) Prove that ϕ is not an isomorphism when V is not finite-dimensional. [Hint: consider the dual of a dual basis.]

3. Let $T : V \to W$ be a linear map between finite-dimensional vector spaces. Let α be a basis for V and let β be a basis for W. Let β^* be the dual basis for W^* and let α^* be the dual basis for V^*. Describe $[T^*]_{\beta^*}^{\alpha^*}$ in terms of $[T]_{\alpha}^{\beta}$.

4. Let V be a vector space and let W be a subspace. Let $\alpha = \{v_1, \ldots, v_k\}$ be a basis for W and extend it to a basis $\beta = \{v_1, \ldots, v_n\}$ for V. Let $\gamma = \{[v_{k+1}], \ldots, [v_n]\}$. Let $T : V \to V$ be a linear operator and let W be a T-invariant subspace. Let $T_W : W \to W$ be the restriction of T to W and let $T_{V/W}$ the be the induced linear operator on V/W.

   (a) Prove that γ is a basis for V/W.

   (b) Explain the relationship among the matrices $[T_W]_\alpha$,$[T_{V/W}]_\gamma$, and $[T]_\beta$.

   (c) Prove that the characteristic polynomial of T is the product of the characterstic polynomials of T_W and $T_{V/W}$.

5. Let V be a vector space and W be a subspace. Let $\phi : V \to V/W$ be the linear map defined by $\phi(v) = [v]$.

   Let X be another vector space and let $T : V \to X$ be a linear map whose null space contains W. Prove that there exists a unique linear map $U : V/W \to X$ such that $T = U \circ \phi$.