MAT247

Definitions (part 2)

Winter 2014

A **basis** of a vector space V is a collection of vectors $\{v_i\}_{i \in I}$ such that for all $v \in V$, there exists a unique collection of scalars $\{a_i\}_{i \in I}$ such that $v = \sum_{i \in I} a_i v_i$ and at most, finitely many a_i are non-zero.

A subset $S \subset V$ is called **linearly independent** if whenever $v_1, ..., v_n \in S$ and there exist $a_1, ..., a_n \in \mathbb{F}$ such that $a_1 v_1 + ... + a_n v_n = 0$, then $a_i = 0$ for all i.

For any V, the **dual space**, Vj, is the set of all linear functionals on V.

Suppose $\{v_i\}_{i \in I}$ is a basis of V. Then define $v_i^* \in V^*$ by

$$v_i^*(v_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

An **equivalence relation on a set**, X, is a collection of ordered pairs $x \sim y$, $x, y \in X$ that satisfies reflexivity, symmetry, and transitivity:

- Reflexivity: $x \sim x$ for all $x \in X$
- Symmetry: $x \sim y$ implies $y \sim x$ for all $x, y \in X$
- Transitivity: if $x \sim y$ and $y \sim z$, then $x \sim z$, for all $x, y, z \in X$.

If $\sim$ is an equivalence relation on X, then for $x \in X$, the **equivalence class** $[x]$ is defined by

$$[x] = \{y \in X : y \sim x\}.$$

$[x] = [y]$ if and only if $x \sim y$. $X/\sim$ denotes the set of all equivalence classes of X.

An **equivalence relation on a vector space**, V, is defined by $v_1 \sim v_2$ if $v_2 - v_1 \in W$, $W \subset V$. Then let $V/W := V/\sim$. We can put a vector space structure on V/W by defining:

- Addition: $[v_1] + [v_2] = [v_1 + v_2]$, $v_1, v_2 \in V$
- Scalar multiplication: $a[v] = [av]$, $a \in \mathbb{F}$, $v \in V$.

Suppose we have $T : V \to V$ and W a T-invariant subspace. Then there is a linear map $T_{v/w} : V/W \to V/W$ defined by $T_{v/w}([v]) = [T(v)]$

For V_1, V_2, W vector spaces, a **bilinear map** $B : V_1 \times V_2 \to W$ is a map satisfying:

- $B(av_1, v_2) = aB(v_1, v_2) = B(v_1, av_2)$ for $v_1 \in V_1$, $v_2 \in V_2$

- $B(v_1+v_1', v_2) = B(v_1,v_2) + B(v_1',v_2)$ and $B(v_1, v_2+v_2') = B(v_1,v_2) + B(v_1,v_2')$ for $v_1, v_1' \in V_1$, $v_2, v_2' \in V_2$.

A **bilinear pairing** is a bilinear map $B : V_1 \times V_2 \to \mathbb{F}$.

A **bilinear form** is a bilinear map $B : V_1 \times V_1 \to \mathbb{F}$.

A bilinear pairing $B : V \times W \to \mathbb{F}$ is called **non-degenerate** if

- for all $v \neq 0$, $v \in V$ there exists $w \in W$ such that $B(v,w) \neq 0$.
- for all $w \neq 0$, $w \in W$ there exists $v \in V$ such that $B(w,v) \neq 0$.

A bilinear $B : V \times W \to \mathbb{F}$ gives rise to two linear maps:

- $\tilde{B} : V \to W^*$ defined by $(\tilde{B}(v))(w) = B(v,w)$
- $\tilde{B} : W \to V^*$ defined by $(\tilde{B}(w))(v) = B(w,v)$

Define $W^{\perp} = \{v \in V^* : \alpha(w) = 0 \; \forall \; w \in W\}$.

If B is a bilinear form on V, then define $W^{\perp,B} = \{v \in V : B(v,w) = 0 \; \forall \; w \in W\}$.

For V and W vector spaces, their **direct sum** is defined by $V \oplus W = \{(v,w) : v \in V, w \in W\}$.

For V and W vector spaces, their **tensor product** is defined by $V \otimes W = \mathbb{F}[V \times W]/Y$, where $\mathbb{F}[V \times W]$ denotes the free vector space on $V \times W$ and Y is defined by:

$$Y = \mathrm{span}((av,w) - a(v,w), (v,aw) - a(v,w), (v_1+v_2,w) - (v_1,w) - (v_2,w), (v, w_1+w_2) - (v,w_1) - (v,w_2))$$

where $a \in \mathbb{F}$, $v_1, v_2, v_3 \in V$, $w_1, w_2, w_3 \in W$. If $v \in V$ and $w \in W$, $v \otimes w := [(v,w)] \in V \otimes W$. If $\{v_1, ..., v_n\}$ is a basis for V and $\{w_1, ..., w_m\}$ is a basis for W, then $\{v_i \otimes v_j\}$ is a basis for $V \otimes W$.

If A is an $n_1 \times m_1$ matrix, B an $n_2 \times m_2$ matrix, then $A \otimes B$ is an $n_1 n_2 \times m_1 m_2$ matrix indexed by: rows labelled by pairs (i_1, i_2), $1 \leq i_1 \leq n_1$ and $1 \leq i_2 \leq n_2$ and columns labelled by pairs (j_1, j_2), $1 \leq j_1 \leq m_1$ and $1 \leq j_2 \leq m_2$, where, $A \otimes B_{(i_1,i_2),(j_1,j_2)} := A_{i_1,j_1} B_{i_2,j_2}$

For V a vector space, the **kth tensor power** of V, is defined by $V^{\otimes k} = V \otimes ... \otimes V$ (k times). Suppose that V has a basis $\{v_1, ..., v_n\}$.

Define a linear map $\tau : V \otimes V \to V \otimes V$ by: $\tau(v_1 \otimes v_2) = v_2 \otimes v_1$ for any $v_1, v_2 \in V$. $\tau^2 = I$.

Define the **symmetric square** of V to be the 1-eigenspace of τ, $\mathrm{Sym}^2 V = \{y \in V^{\otimes 2} : \tau(y) = y\}$. $\mathrm{Sym}^2 V$ has a basis $\{v_i \otimes v_i \; \forall \; i, v_i \otimes v_j + v_j \otimes v_i \; 1 \leq i < j \leq n\}$.

Define the **exterior square** of V to be the -1-eigenspace of τ, $\bigwedge^2 V = \{y \in V^{\otimes 2} : \tau(y) = -y\}$. $\bigwedge^2 V$ has a basis $\{v_i \otimes v_j - v_j \otimes v_i \; 1 \leq i < j \leq n\}$.

Let V be a vector space. We can consider $V^{\otimes 1}$, $V^{\otimes 2}$, ... Then we can define the **tensor algebra** $TV = \bigoplus_{k=0}^{\infty} V^{\otimes k}$.

A **transposition** σ is a permuation which just switches two elements. So there exist $i \neq j$ with $\sigma(i) = j$, $\sigma(j) = i$ and $\sigma(l) = l$ for all $l \neq i, j$.

Recall **sign**, $\mathrm{sign} : S_k \to \{1, -1\}$. The sign function has the following properties:

- $\text{sign}(\sigma_1\sigma_2) = \text{sign}(\sigma_1)\text{sign}(\sigma_2)$
- σ is a transposition, then $\text{sign}(\sigma) = -1$

Define the **symmetric power** of V, $\text{Sym}^k V = \{y \in V^{\otimes k} : \sigma(y) = y \ \forall\ \sigma \in S_k\}$. Define $v_1 \cdot v_2 \cdot ... \cdot v_k = \sum_{\sigma \in S_k} v_{\sigma(1)} \otimes ... \otimes v_{\sigma(k)}$. Then $\sigma(v_1 \cdot ... \cdot v_k) = v_1 \cdot ... \cdot v_k$. Let $\{v_1, ..., v_k\}$ be a basis for V. Then $\{v_{i_1} \cdot ... \cdot v_{i_k} : 1 \leq i_1 \leq ... \leq i_k \leq n\}$ forms a basis for $\text{Sym}^k V$.

Define the **exterior power** or **wedge power** of V, $\bigwedge^k V = \{y \in V^{\otimes k} : \sigma(y) = sign(\sigma)y \ \forall\ \sigma \in S_k\}$. Define $v_1 \wedge v_2 \wedge ... \wedge v_k = \sum_{\sigma \in S_k} sign(\sigma) v_{\sigma(1)} \otimes ... \otimes v_{\sigma(k)}$. Let $\{v_1, ..., v_k\}$ be a basis for V. Then $\{v_{i_1} \wedge ... \wedge v_{i_k} : i_1 < ... < i_k\}$ forms a basis for $\bigwedge^k V$.

Let $T : V \to W$. Then for all $k \geq 0$, we can define:

- $T^{\otimes k} : V^{\otimes k} \to W^{\otimes k}$ by $T^{\otimes k}(v_1 \otimes ... \otimes v_k) = Tv_1 \otimes ... \otimes Tv_k$.
- $\text{Sym}^k T : \text{Sym}^k V \to \text{Sym}^k W$ by $\text{Sym}^k T(v_1 \cdot ... \cdot v_k) = Tv_1 \cdot ... \cdot Tv_k$.
- $\bigwedge^k T : \bigwedge^k V \to \bigwedge^k W$ by $\bigwedge^k T(v_1 \wedge ... \wedge v_k) = Tv_1 \wedge ... \wedge Tv_k$. Note: $\bigwedge^k T = \det T$

Let A be a square matrix. Define the **trace** of A by $\text{tr}(A) = \sum_{i=1}^n A_{i,i}$, the sum of the diagonal entries of A. The trace of A is also the coefficient of x in $\det(xI - A)$.