ASSIGNMENT 8
DUE APRIL 5

(1) Fix a finite group G and a field k.
   (a) Suppose that $\phi : G \to k^\times$ be a non-trivial group homomorphism. Prove that $\sum_{g \in G} \phi(g) = 0$.
   (b) Let $p = char k$ and assume that $p \mid |G|$. Prove that kG is not a semisimple algebra by showing that the regular representation of kG has a subrepresentation without complement. (Hint: use part (a).)

(2) A division ring (or skew field) is a ring in which every element has a multiplicative inverse (i.e. it is like a field, but we don't demand that multiplication is commutative).
   (a) Let V be a finite-dimensional irreducible representation of a k-algebra A, where k is not necessarily algebraically closed. Prove that $Hom_A(V, V)$ is a division ring.
   (b) Find an example of a finite group G and an irreducible representation V of G over $\mathbb{R}$ such that $Hom_G(V, V)$ is isomorphic to the division ring of quaternions $\mathbb{H}$.

(3) Fix a field k. Consider the following 3-dimensional k-algebra A. It has a k-basis given by x, t, z and multiplication given by

$$x^2 = x, xt = t, tz = t, z^2 = z$$

and all other products are 0. (The identity element is $x + z$.)
   (a) Prove that a representation of A is the same thing as a pair of k-vector spaces X, Z and a linear map $T : X \to Z$.
   (b) Show that A is not semisimple.
   (c) Find all indecomposable representations of A.

(4) Prove that the number of 1-dimensional representations of a finite group G over $\mathbb{C}$ equals $|G|/|G'|$ where G' denotes the commutator subgroup of G.

(5) Find the character table of the group S_4.

(6) Fix a prime p and a field k (not necessarily algebraically closed) of characteristic p.

   Let G be a p-group (i.e. a group of order p^n for some n). Prove that the only irreducible representation of G is the trivial representation. (Hint: Consider an element $g \in Z(G)$ of order p and show that g must act by the identity on an irrep V.)