ASSIGNMENT 7
DUE THURSDAY MARCH 22

(1) The goal of this exercise is to prove (and then apply) the following result, known as the going-up theorem.

Theorem 1. *Let $R \subset S$ be two rings such that S is finitely generated as an R-module. If P is a prime ideal in R, then there exists a prime ideal Q in S such that $Q \cap R = P$.*

(Actually the theorem holds (with almost the same proof) in the more general case that S is integral over R.)

(a) Assume that R is a local ring and P is its unique maximal ideal. Use Nakayama's lemma to prove the going-up theorem in this special case.
(b) Use localization at P to deduce the general case of the going-up theorem from the above special case.
(c) Suppose that X and Y are algebraic varieties over an algebraic closed field. Let $\phi : X \to Y$ be a morphism such that $\phi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ is injective and $\mathcal{O}(X)$ is finitely generated as a $\phi^*(\mathcal{O}(Y))$-module. Use the going-up theorem to prove that ϕ is surjective.
(d) Find an example of a morphism $\phi : X \to Y$ such that ϕ^* is injective, but ϕ is not surjective.

(2) Let k be an algebraically closed field and let X be an affine variety over k. Show that giving a k-algebra homomorphism $\mathcal{O}(X) \to k[x]/x^2$ is equivalent to giving a point $a \in X$ and an element $v \in T_aX$.

(3) (a) Let M be an R-module. Prove that if $M_P = 0$ for all prime ideals P, then $M = 0$.
(b) Let $\phi : M \to N$ be a morphism of R-modules. Prove that ϕ is injective (resp. surjective) if and only if $\phi_P : M_P \to N_P$ is injective (resp. surjective) for all prime ideals P.

(4) Let k be a field and consider the ideal $I = \langle xy, y^2 \rangle \subset k[x, y]$. Let $M = k[x, y]/I$. Find $Ass(M)$ and $Supp(M)$.